e-ISSN: 3027-0650
pp. 5-13, November 13-16, 2023

A Continuous Hybrid Scheme for Initial Value Problem of Third Order Ordinary Differential Equations.

F. L. Joseph
{folakelois@gmail.com}
Department of Mathematical Sciences, Bingham University, Karu,
Nasarawa State, Nigeria.

Abstract
In this work, we focus on development of solution for initial value problems of third order ordinary differential equations using a new class of constructed orthogonal polynomial of weight function $w(x) = x$ valid in the interval [0,1],as basis function for the development of continuous hybrid scheme in a collocation and interpolation technique. The method was analyzed to investigate the basic properties, from the findings it shows that the method is accurate and convergent. Three examples were solved, the results obtained when compared with existing method are favourable.

Keyword: Orthogonal polynomial, Hybrid, Interpolation, Collocation, Block Method

1 Introduction

Initial Value Problems (IVPs) of third order ordinary differential equations (ODEs) of the form

$$y''' = f(x, y, y', y''); y(a) = \alpha, y'(a) = \beta, y''(a) = \gamma \qquad (1)$$

where f is continuous in $[a, b]$ arises in many area of physical problems.
Some of these problems have no analytical solution, thereby numerical schemes are developed to solve the problems. Milne (1953), proposed Block method for ODEs. Many researchers used different orthogonal polynomials as the basis function to solve the problems numerically. Chebyshev orthogonal polynomial was used by Lancsos(1983)also Tanner (1979) and Dahlguist (1979). Adeniyi, Alabi and Folaranmi (2008), Adeyefa, Akinola, Folaranmi and Owolabi (2016), Joseph, Adeniyi and Adeyefa(2018), all of these researchers constructed orthogonal polynomials in certain interval for different weight functions. In this work, an orthogonal polynomial constructed for the interval [0,1] with respect to the weight function $w(x) = x$ is adopted to solve third order ODEs for the Initial Value Problem (1).

2 Construction of Orthogonal Polynomials

Let $\{\phi_n(x)\}$ be a class of orthogonal polynomials defined by

$$\phi_n(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \qquad (2)$$

The required conditions are as follows:

$$\phi_n(1) = 1 \qquad (3)$$

$$< \phi_m(x), \phi_n(x) > = 0, \; m \neq n \qquad (4)$$

This class of orthogonal polynomials valid in the interval [0,1] and weight function

$$w(x) = x.$$

Let $w(x) = x$ and $[a, b] = [0,1]$ in (2) - (4).
when $n = 0$, we have

$\phi_0(x) = C_0^{(0)}$ and
$\phi_0(1) = 1 = C_0^{(0)}$ giving $\phi_0(x) = 1$
For $n = 1$, we have

$$\phi_1(x) = C_0^{(1)} + C_1^{(1)}x$$

$$\therefore\ \phi_1(1) = C_0^{(1)} + C_1^{(1)} = 1 \tag{5}$$

$$\langle \phi_0(x), \phi_1(x) \rangle = \int_0^1 x\left(C_0^{(1)} + C_1^{(1)}x\right)dx = 0$$

That is,

$$\frac{1}{2}C_0^{(1)} + \frac{1}{3}C_1^{(1)} = 0 \tag{6}$$

The solution of (5)-(6) yields

$$C_0^{(1)} = -2,\ C_1^{(1)} = 3$$

Hence,
$\phi_1(x) = -2 + 3x$ or $\phi_1(x) = 3x - 2$
For $n = 2$, we have

$$\phi_2(x) = C_0^{(2)} + C_1^{(2)}x + C_2^{(2)}x^2$$

$$\therefore\ \phi_2(1) = C_0^{(2)} + C_1^{(2)} + C_2^{(2)} = 1 \tag{7}$$

$$< \phi_0(x), \phi_2(x) >= \frac{1}{2}C_0^{(2)} + \frac{1}{3}C_1^{(2)} + \frac{1}{4}C_2^{(2)} = 0 \tag{8}$$

$$< \phi_0(x), \phi_2(x) >= \frac{1}{12}C_1^{(2)} + \frac{1}{10}C_2^{(2)} = 0 \tag{9}$$

From these equations, we get

$$C_0^{(2)} = 3,\ C_1^{(2)} = -12,\ C_2^{(2)} = 10$$

Hence,

$$\phi_2(x) = 3 - 12x + 10x^2$$

Similarly, we obtain more polynomials to give the following collection:

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= 3x - 2 \\
\phi_2(x) &= 10x^2 - 12x + 3 \\
\phi_3(x) &= 35x^3 - 60x^2 + 30x - 4 \\
\phi_4(x) &= 126x^4 - 280x^3 + 210x^2 - 60x + 5 \\
\phi_5(x) &= 462x^5 - 1260x^4 + 1260x^3 - 560x^2 + 105x - 6 \\
\phi_6(x) &= 1716x^6 - 5544x^5 + 6930x^4 - 4200x^3 + 1260x^2 - 168x + 7 \\
\phi_7(x) &= 6435x^7 - 24024x^6 + 36036x^5 + 27720x^4 + 11550x^3 - 2520x^2 + 252
\end{aligned} \tag{10}$$

2.1 Two-step Method with $x_{n+\frac{2}{3}}$ as the Off-step Point
The analytical solution of (1) is approximated via experimental solution of the form:

$$Y(x) = \sum_{j=0}^{r+s-1} a_j\phi_j(x) \tag{11}$$

where $x \in [a, b]$, r and s are the number of collocation and interpolation points respectively. The function $\phi_j(x)$ is the j^{th} degree orthogonal polynomial valid in the range of integration of $[a, b]$. The third derivative of (11) is given by

$$y'''(x) = \sum_{j=0}^{r+s-1} a_j \phi_j'''(x) = f(x, y, y', y'') \tag{12}$$

To estimate the solution of problem (1), we interpolation at least three times. Equation (11) is interpolated at $(xn + s)$ points, and equation (12) is collocated at $(xn + r)$ points, yielding a system of equations to be solved using the Gaussian elimination method.

We will use hybrid approach to apply this concept.

Here, let $x_{n+\frac{2}{3}}$ be the off-step point. Equation (11) is interpolated at $x = x_{n+s}, s = 0, \frac{2}{3}$ and 1; (12) is collocated at $x = x_{n+r}, r = 0, \frac{2}{3} 1$ and 2. This leads to the system of equations:

$$\begin{bmatrix} 1 & -5 & 25 & -129 & 681 & -3653 & 19825 \\ 1 & -3 & \frac{73}{9} & \frac{-593}{27} & \frac{1627}{27} & \frac{-13555}{81} & \frac{11732}{25} \\ 1 & -2 & 3 & -4 & 5 & -6 & 7 \\ 0 & 0 & 0 & 210 & -4704 & 65520 & -730080 \\ 0 & 0 & 0 & 210 & -2688 & 20720 & \frac{-375680}{3} \\ 0 & 0 & 0 & 210 & -1680 & 7560 & -25200 \\ 0 & 0 & 0 & 210 & 1344 & 5040 & 14400 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ h^3 f_n \\ h^3 f_{n+\frac{2}{3}} \\ h^3 f_{n+1} \\ h^3 f_{n+2} \end{bmatrix} \tag{13}$$

Solving system (12) to obtain the values of the unknown parameters $a_j, j = 0(1)6$ yielded:

$$\begin{aligned}
a_0 &= \frac{13}{12} y_n - \frac{21}{4} y_{n+\frac{2}{3}} + \frac{31}{6} y_{n+1} + \frac{56h^3}{6051} f_n + \frac{173h^3}{2366} f_{n+\frac{2}{3}} + \frac{593h^3}{4656} f_{n+1} + \frac{35h^3}{6892} f \\
a_1 &= \frac{23}{30} y_n - \frac{33}{10} y_{n+\frac{2}{3}} + \frac{38}{15} y_{n+1} + \frac{27h^3}{3238} f_n + \frac{173h^3}{4746} f_{n+\frac{2}{3}} + \frac{227h^3}{1843} f_{n+1} + \frac{53h^3}{8667} f \\
a_2 &= \frac{3}{20} y_n - \frac{9}{20} y_{n+\frac{2}{3}} + \frac{3}{10} y_{n+1} + \frac{110h^3}{33159} f_n - \frac{34h^3}{4863} f_{n+\frac{2}{3}} + \frac{31h^3}{613} f_{n+1} + \frac{13h^3}{3336} f_n \\
a_3 &= h^3 \left(\frac{29}{41580} f_n - \frac{17}{3080} f_{n+\frac{2}{3}} + \frac{4}{495} f_{n+1} + \frac{25}{16632} f_{n+2} \right) \\
a_4 &= h^3 \left(\frac{1}{38016} f_n - \frac{1}{19712} f_{n+\frac{2}{3}} - \frac{1}{3168} f_{n+1} + \frac{13}{38226} f_{n+2} \right) \\
a_5 &= -h^3 \left(\frac{6}{23374} f_n - \frac{9}{45760} f_{n+\frac{2}{3}} + \frac{19}{90090} f_{n+1} - \frac{1}{250685} f_{n+2} \right) \\
a_6 &= -h^3 \left(\frac{1}{274560} f_n - \frac{3}{183040} f_{n+\frac{2}{3}} + \frac{1}{68640} f_{n+1} - \frac{1}{549120} f_{n+2} \right)
\end{aligned} \tag{14}$$

Substituting (14) in (11) gives a continuous implicit two-step method in the form

$$\bar{y}(x) = \sum_{j=0}^{1} \alpha_j(x) y_{n+j} + \alpha_{\frac{2}{3}}(x) y_{n+\frac{2}{3}} + h^3 \left(\sum_{j=0}^{2} \beta_j(x) f_{n+j} + \beta_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} \right) \tag{15}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients. From (15) the parameters $\alpha_j(x)$ and $\beta_j(x)$ are given by:

$$\alpha_0(t) = \frac{3}{2}t^2 + \frac{t}{2}$$
$$\alpha_{\frac{2}{3}}(t) = -\frac{9t^2}{2} - \frac{9t}{2}$$
$$\alpha_1(t) = 3t^2 + 4t + 1$$
$$\beta_0(t) = -h^3\left(\frac{t^6}{160} - \frac{t^5}{120} - \frac{t^4}{96} + \frac{t^3}{235058424339189180} - \frac{43t^2}{6480} - \frac{2260t}{915301} + \frac{1}{300349807825007740}\right)$$
$$\beta_{\frac{2}{3}}(t) = h^3\left(\frac{9t^6}{320} + \frac{t^5}{12688236664456307} - \frac{9t^4}{64} + \frac{16748472397082323}{163} + \frac{29t^2}{180} - + \frac{11866908084007885}{14}\right) \quad (16)$$
$$\beta_1(t) = -h^3\left(\frac{t^6}{40} - \frac{t^5}{60} - \frac{t^4}{8} + \frac{t^3}{6} - \frac{22t^2}{405} - \frac{247t}{57163} + \frac{t^3}{30264189495929732}\right)$$
$$\beta_2(t) = h^3\left(\frac{t^6}{320} + \frac{t^5}{120} + \frac{t^4}{192} - \frac{1}{3695570313586332200} + \frac{18t^2}{116639} + \frac{18t}{116639} - \frac{1}{373665233501228290}\right)$$

By evaluating (15) at x_{n+2}, the main method is obtained as

$$y_{n+2} = 2y_n - 9y_{n+\frac{2}{3}} + 8y_{n+1} + h^3\left(\frac{7}{324}f_n + \frac{7}{72}f_{n+\frac{2}{3}} + \frac{25}{81}f_{n+1} + \frac{11}{648}f_{n+2}\right) \quad (17)$$

Differentiate (15), to get the continuous coefficients:

$$\alpha_0'(t) = \frac{3t + \frac{1}{2}}{h}$$
$$\alpha_{\frac{2}{3}}(t) = \frac{-\left(9t + \frac{9}{2}\right)}{h}$$
$$\alpha_1'(t) = \frac{(6t + 4)}{h}$$
$$\beta_0'(t) = -h^2\left(\frac{3t^5}{80} - \frac{t^4}{24} - \frac{t^3}{24} + \frac{t^2}{78352808113063056} - \frac{43t}{3240} - \frac{2260}{9153301}\right)$$
$$\beta_{\frac{2}{3}}'(t) = h^2\left(\frac{27t^5}{760} + \frac{t^4}{24} - \frac{9t^3}{16} + \frac{29t}{558282412360774700} + \frac{7}{90} + \frac{7}{144}\right) \quad (18)$$
$$\beta_1'(t) = h^2\left(\frac{-(3t^5)}{20} - \frac{t^4}{12} + \frac{t^3}{2} + \frac{t^2}{2} + \frac{2201759380413025t}{20266198323167232} + \frac{9121851463349}{9153301}\right)$$
$$\beta_2'(t) = h^2\left(\frac{3t^5}{160} + \frac{t^4}{24} + \frac{t^3}{48} - \frac{t^2}{1231856771195444} + \frac{35t}{113399} + \frac{18}{116639}\right)$$

The second derivatives of continuous functions (15) yield the following coefficient

$$
\begin{aligned}
\alpha_0''(t) &= \frac{3}{h^2}\\
\alpha_{\frac{2}{3}}''(t) &= \frac{-9}{h^2}\\
\alpha_1''(t) &= \frac{6}{h^2}\\
\beta_0''(t) &= -h\left(\frac{3t^4}{16}-\frac{t^3}{6}-\frac{t^2}{8}+\frac{t}{3917640405653 1528}-\frac{43}{3240}\right)\\
\beta_{\frac{2}{3}}''(t) &= h\left(\frac{27t^4}{32}+\frac{t^3}{634411833222815230}-\frac{27t^2}{16}+\frac{t}{2791412066180387300}+ \cdots\right)\\
\beta_1''(t) &= -h\left(\frac{3t^4}{4}-\frac{t^3}{3}+\frac{3t^2}{2}+t+\frac{44}{405}\right)\\
\beta_2''(t) &= h\left(\frac{3t^4}{32}+\frac{t^3}{6}+\frac{t^2}{16}-\frac{t}{615928385597721980}+\frac{35}{113399}\right)
\end{aligned}
\tag{19}
$$

The additional methods to be coupled with the main method (17) are obtained by evaluating the first and second derivatives of (15) at x_n, $x_{n+\frac{2}{3}}$, x_{n+1} and x_{n+2} respectively to obtain:

$$
hy_n' + \frac{5}{2}y_n - \frac{9}{2}y_{n+\frac{2}{3}} + 2y_{n+1} = h^3\left(\frac{173f_n}{6480}+\frac{173f_{n+\frac{2}{3}}}{1440}-\frac{61f_{n+1}}{1620}+\frac{5f_{n+2}}{2592}\right) \tag{20}
$$

$$
hy_{n+\frac{2}{3}}' + \frac{y_n}{2} + \frac{3}{2}y_{n+\frac{2}{3}} - 2y_{n+1} = h^3\left(\frac{-11f_n}{3888}-\frac{167f_{n+\frac{2}{3}}}{4320}-\frac{23f_{n+1}}{4860}-\frac{29f_{n+2}}{102502}\right) \tag{21}
$$

$$
hy_{n+1}' - \frac{y_n}{2} + \frac{9}{2}y_{n+\frac{2}{3}} - 4y_{n+1} = h^3\left(\frac{2260f_n}{915301}+\frac{7f_{n+\frac{2}{3}}}{144}+\frac{247f_{n+1}}{57163}+\frac{18f_{n+2}}{116639}\right) \tag{22}
$$

$$
hy_{n+2}' - \frac{7}{2}y_n + \frac{27}{2}y_{n+\frac{2}{3}} - 10y_{n+1} = h^3\left(\frac{133f_n}{2160}-\frac{1312f_{n+\frac{2}{3}}}{57251}+\frac{95f_{n+1}}{108}+\frac{353f_{n+2}}{4320}\right) \tag{23}
$$

$$
h^2y_n'' - 3y_n + 9y_{n+\frac{2}{3}} - 6y_{n+1} = h^3\left(-\frac{527f_n}{2441}-\frac{751f_{n+\frac{2}{3}}}{1440}+\frac{311f_{n+1}}{1620}-\frac{131f_{n+2}}{12960}\right) \tag{24}
$$

$$h^2 y''_{n+\frac{2}{3}} - 3y_n + 9y_{n+\frac{2}{3}} - 6y_{n+1} = h^3\left(\frac{121f_n}{6480} + \frac{209f_{n+\frac{2}{3}}}{1440} - \frac{89f_{n+1}}{1620} + \frac{77f_{n+2}}{34411}\right) \tag{25}$$

$$h^2 y''_{n+1} - 3y_n + 9y_{n+\frac{2}{3}} - 6y_{n+1} = h^3\left(\frac{43f_n}{3240} + \frac{29f_{n+\frac{2}{3}}}{90} + \frac{44f_{n+1}}{405} + \frac{35f_{n+2}}{113399}\right) \tag{26}$$

$$h^2 y''_{n+2} - 3y_n + 9y_{n+\frac{2}{3}} - 6y_{n+1} = h^3\left(\frac{761f_n}{6480} - \frac{751f_{n+\frac{2}{3}}}{1440} - \frac{1115f_{n+1}}{731} + \frac{469f_{n+2}}{1451}\right) \tag{27}$$

Equations (17) and (20) - (27) are solved using Shampine and Watts (1969) block formula defined as

$$Ay_m = hBF(y_m) + E_{y_n} + hDf_n \tag{28}$$

$$A = \begin{pmatrix} 9 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{9}{2} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{3}{2} & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{9}{2} & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{27}{2} & -10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 9 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} B$$

$$= \begin{pmatrix} 7/72 & 25/81 & 11/648 \\ 173/1440 & -61/1620 & 5/2592 \\ -167/4320 & 23/4860 & -29/102502 \\ 7/144 & 247/57163 & 18/116639 \\ -1312/57251 & 95/108 & 353/4320 \\ -751/1440 & 311/1620 & -131/12960 \\ 209/1440 & -89/1620 & 77/34411 \\ 29/90 & 44/405 & 35/113399 \\ -751/1440 & 1115/731 & 469/1451 \end{pmatrix}$$

$$D = \begin{pmatrix} 7/324 \\ 173/6480 \\ -11/3888 \\ 2260/915301 \\ 133/2160 \\ -527/2441 \\ 121/6480 \\ 43/3240 \\ 761/6480 \end{pmatrix} \quad E = \begin{pmatrix} 2 & 0 & 0 \\ -5/2 & -1 & 0 \\ -1/2 & 0 & 0 \\ 1/2 & 0 & 0 \\ 7/2 & 0 & 0 \\ 3 & 0 & -1 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Substituting A, B, D and E into equation (28) the following equations are obtained:

$$y_{n+\frac{2}{3}} = y_n + \frac{2}{3}y_n' + \frac{2}{9}y_n'' + \frac{22}{729}f_n + \frac{29}{810}f_{n+\frac{2}{3}} - \frac{64}{3645}f_{n+1} + \frac{7}{7290}f_{n+2} \tag{29}$$

$$y_{n+1} = y_n + y_n' + \frac{1}{2}y_n'' + \frac{13}{160}f_n + \frac{9}{64}f_{n+\frac{2}{3}} - \frac{7}{120}f_{n+1} + \frac{1}{320}f_{n+2} \tag{30}$$

$$y_{n+2} = y_n + 2y_n' + 2y_n'' + \frac{2}{5}f_n + \frac{9}{10}f_{n+\frac{2}{3}} + \frac{1}{30}f_{n+2} \tag{31}$$

$$y'_{n+\frac{2}{3}} = y_n' + \frac{2}{3}y_n'' + \frac{139}{1215}f_n + \frac{17}{90}f_{n+\frac{2}{3}} - \frac{104}{1215}f_{n+1} + \frac{11}{2430}f_{n+2} \tag{32}$$

$$y'_{n+1} = y_n' + y_n^n + \frac{23}{120}f_n + \frac{9}{20}f_{n+\frac{2}{3}} - \frac{3}{20}f_{n+1} + \frac{1}{120}f_{n+2} \tag{33}$$

$$y'_{n+2} = y_n' + 2y_n'' + \frac{7}{15}f_n + \frac{9}{10}f_{n+\frac{2}{3}} + \frac{8}{15}f_{n+1} + \frac{1}{10}f_{n+2} \tag{34}$$

$$y''_{n+\frac{2}{3}} = y_n'' + \frac{19}{81}f_n + \frac{2}{3}f_{n+\frac{2}{3}} - \frac{20}{81}f_{n+1} + \frac{1}{81}f_{n+2} \tag{35}$$

$$y''_{n+1} = y_n'' + \frac{11}{48}f_n + \frac{27}{32}f_{n+\frac{2}{3}} - \frac{1}{12}f_{n+1} + \frac{1}{96}f_{n+2} \tag{36}$$

$$y''_{n+2} = y_n'' + \frac{1}{3}f_n + \frac{4}{3}f_{n+1} + \frac{1}{3}f_{n+2} \tag{37}$$

2.2 Analysis of the Methods

The basic properties are order, error constant, zero stability and consistency.
The main methods derived are discrete schemes belonging to the class of LMMs of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{38}$$

Following Futunla (1988) and Lambert (1973), we define the Local Truncation Error (LTE) associated with (38) by difference operator;

$$L[y(x):h] = \sum_{j=0}^{k} \left[\alpha_j y(x_n + jh) - h^3\beta_j f(x_n + jh)\right] \tag{39}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Expanding (3) in Taylor's Series about the point x, we obtain the expression

$$L[y(x):h] = c_o y(x) + c_1 h y'(x) + \cdots c_{p+3} h^{p+3} y^{p+3}(x) \tag{40}$$

where the $c_o, c_1, c_2 \ldots c_p \ldots c_{p+3}$ are obtained

$$c_0 = \sum_{j=0}^{k} \alpha_j \tag{41}$$

$$c_1 = \sum_{j=1}^{k} j\alpha_j \tag{42}$$

$$c_3 = \frac{1}{3!}\sum_{j=1}^{k} j^3\alpha_j \tag{43}$$

$$c_q = \frac{1}{q!}\left[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)(q-2)(q-3)\sum_{j=1}^{k} \beta_j j^{q-3}\right] \tag{44}$$

In the sense of Lambert (1973), equation (38) is of order p if $c_o = c_1 = c_2 = c_2 = \cdots c_p = c_{p+1} = c_{p+2} = 0$ and $c_{p+3} \neq 0$. The $c_{p+3} \neq 0$ is called the error constant and $c_{\mathrm{p+3}}h^{\mathrm{p+3}}y^{\mathrm{p+3}}(x_n)$ is the Principal Local truncation error at the point x_n. The equation (17) is of order $\mathrm{p} = 4$ and error constants $C_{p+3} = -\frac{31}{29160}$

2.2.1 Zero stability

The LMM (1) is said to be Zero-stable if no root of the first characteristic polynomial $\rho(R)$ has modulus greater than one and if and only if every root of modulus one has multiplicity not greater than the order of the differential equation.

2.2.2 Consistency

The LMM is said to be consistent if it has order $p \geq 1$ and the first and second characteristic polynomials which are defined respectively, as

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \tag{45}$$

and

$$\sigma(z) = \sum_{j=0}^{k} \beta_j z^j \tag{46}$$

where z is the principal root, satisfy the following conditions:

$$\sum_{j=0}^{k} \alpha_j = 0 \tag{47}$$

$$\rho(1) = \rho'(1) = 0 \tag{48}$$

And

$$\rho'''(1) = 3 \cdot \sigma(1) \tag{49}$$

(Henrichi, 1962)

The scheme (18) is of order $\rho = 4 > 1$ and they have been investigated to satisfy conditions (I)-(III) of Definition (47) -(49). Hence, the scheme is consistent.

2.2.3 Convergence
According to the theorem of Dahlguist, the necessary and sufficient condition for an LMM to be convergent, is that, it is consistent and zero-stable.
The methods satisfy the two conditions stated in Definition (47) $-(49)$ and hence the method is convergent.

2.2.4 Zero stability of the Method.
To analyze the Zero-stability of the method, equations (29)-(37) is represented in block form below:

$$A^0 y_m = hBF(y_m) + A' y_n hDf_n$$

where h is a fixed mesh size within a block.
The zero stability of equations (29)-(37) gives

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^r = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 29/810 & -64/3645 & 7/7290 \\ 9/64 & -7/120 & 1/320 \\ 9/10 & 0 & 1/30 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 22/729 \\ 0 & 0 & 13/160 \\ 0 & 0 & 2/5 \end{bmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = \det(RA^0 - A') \qquad (50)$$

Substituting A^0 and A' in equation (50) and solving for R, the values of R are obtained as 0 and 1. According to Fatunla (1988,1991), the block method equations (29)-(37) are zero-stable, since from (50), $\rho(R) = 0$, satisfy $|R_j| \leq 1, j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed three.

2.3 Region of Absolute Stability (RAS)
For the Two-step with Off-step Point $\frac{2}{3}$, we have

$$y_{n+2} + 9y_{n+\frac{2}{3}} - 8y_{n+1} - 2y_n = \frac{h^3}{648}\left(14f_n + 63f_{n+\frac{3}{3}} + 200f_{n+1} + 11f_{n+2}\right)$$

$$\bar{h}(z) = \frac{648\left(z^2 + 9z^{\frac{2}{3}} - 8z - 2\right)}{63z^{\frac{2}{3}} + 200z + 11z^2 + 14}$$

$$\bar{h}(\theta) = \frac{648e^{i2\theta} + 9e^{i\frac{2}{3}\theta} - 8e^{i\theta} - 2}{11e^{i2\theta} + 63e^{i\frac{2}{3}\theta} + 200e^{i\theta} + 14}$$

The RAS is shown in the figure below

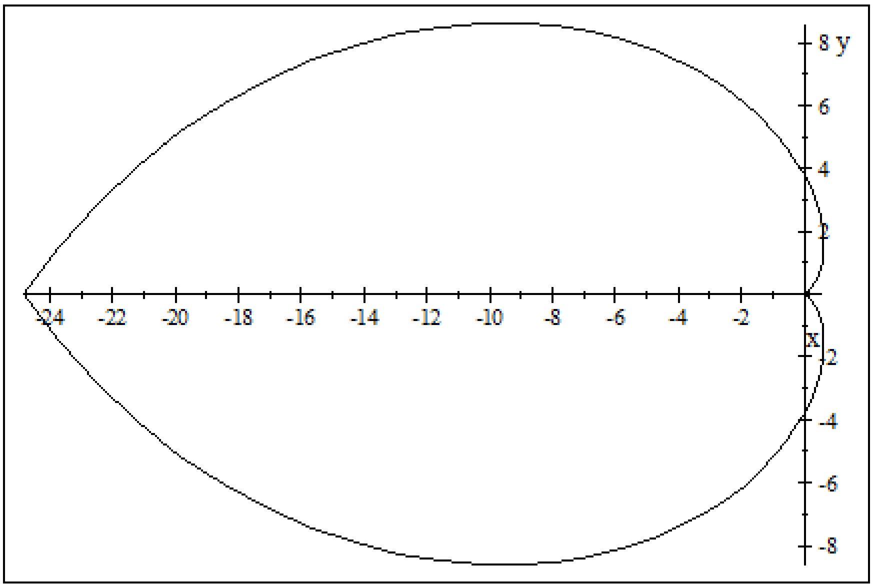

Figure 1: Region of Absolute Stability for Two-step with Off-step Point $\frac{2}{3}$

3 Application of the Method

Three problems characterized by different features will be considered in this section.

Problem 1
The highly nonlinear problem

$$y''' + e^{-y} - 3e^{-2y} + 2e^{-3y} = 0$$
$$y(0) = \ln 2, y'(0) = \frac{1}{2}, y''(0) = \frac{1}{4}$$

sourced from Muhammed (2016) whose analytic solution is

$$y(x) = \ln(e^{x} + 1)$$

was considered with $h = 0.1$,

Problem 2

The nonlinear application problem called Blasius Equation

$$2y''' + yy'' = 0$$
$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 1$$

and sourced from Adesanya et al (2014) was solved here with $h = 0.1$.

Problem 3
An Application Problem (Nonlinear Genesio Equation)
Considered, the nonlinear chaotic system from Genesio and Tesi (1992)

$$x''''(t) + Ax''(t) + Bx'(t) = x^2(t) - Cx(t)$$
$$x(0) = 0.2,\ x'(0) = -0.3,\ x^n(0) = 0.1,\ t \in [0,1]$$

where $A = 1.2, B = 2.29$ and $C = 6$ are positive constants that satisfied $AB < C$ for the existence of the solution.

3 Tables of Results

Table 1: Results for Problem 1

x	Exact solution	Two-steps with $v = \frac{2}{3}$
0.1	0.744396660073572	0.744396660068558
0.2	0.798138869381592	0.798138869344556
0.3	0.854355244468526	0.854355244286741
0.4	0.913015252399952	0.913015251874836
0.5	0.974076984180107	0.974076982907236
0.6	1.037487950485890	1.037487947917230
0.7	1.103186048885460	1.103186044201130
0.8	1.171100665947780	1.171100658162400
0.9	1.241153874732090	1.241153862586050
1.0	1.313261686336555	1.313261669600100

Table 2: Results for Problem 2

x	Analytical solution	two-step method with $v = \frac{2}{3}$
0.1	0.0049999551874560	0.004999958347116
0.2	0.0199986590802381	0.019998666938153
0.3	0.04498987410259470	0.044989879952218
0.4	0.0799573773516761	0.079957379380653
0.5	0.1248700476465370	0.124870060961250
0.6	0.179677126361217	0.179677148374987
0.7	0.2443036129003850	0.244303630177998
0.8	0.3186459794646740	0.318646031552947
0.9	0.4025686062131340	0.402568655792802
10	0.4959003376293370	0.495900435482723

Table 3: Results for Problem 3

X	Analytical solution	two-step method with $v = \frac{2}{3}$ as the Off-step point
0.1	0.170440346269364	0.170440346869070
0.2	0.141582173138664	0.141582171130925
0.3	0.113282963581607	0.113282958227538
0.4	0.085554524922736	0.085554520377859
0.5	0.058543682864593	0.058543674213629
0.6	0.032510877478247	0.032510862344549
0.7	0.007806854082744	0.007806836744866
0.8	−0.015152336804258	−0.015152348568936
0.9	−0.035911645118586	−0.035911639463907
10	−0.054004107797261	−0.054004072464382

4 Tables of Errors

Table 4: Error for Problem 1

x	Error in Anake (2013)	Two-steps $v = \frac{2}{3}$
0.1	1.608800×10^{-9}	5.014000×10^{-12}
0.2	1.038700×10^{-8}	3.703600×10^{-11}
0.3	2.957200×10^{-8}	1.817850×10^{-10}
0.4	2.314700×10^{-7}	5.251160×10^{-10}
0.5	4.542000×10^{-7}	1.272871×10^{-9}
0.6	1.474600×10^{-6}	2.568660×10^{-9}
0.7	2.873400×10^{-6}	4.684330×10^{-9}
0.8	4.682600×10^{-6}	7.785380×10^{-9}
0.9	6.921700×10^{-6}	1.214604×10^{-8}
1.0	9.597400×10^{-6}	1.673646×10^{-8}

Tables of Errors

Table 5: Error for Problem 2

X	Two-steps with $v = \frac{2}{3}$	Error in Anake Block Algorithm
0.1	$3.159660000 \times 10^{-9}$	4.2730000×10^{-8}
0.2	$7.8579149000 \times 10^{-9}$	1.2075900×10^{-6}
0.3	$5.849623300 \times 10^{-9}$	8.6071900×10^{-6}
0.4	$2.028976900 \times 10^{-9}$	$3.40900400 \times 10^{-5}$
0.5	$1.331471300 \times 10^{-8}$	9.7406800×10^{-5}
0.6	$2.201377000 \times 10^{-8}$	2.2571100×10^{-4}
0.7	$1.727761300 \times 10^{-8}$	4.5145470×10^{-4}
0.8	$5.208827300 \times 10^{-8}$	8.084729×10^{-4}
0.9	$4.9579668000 \times 10^{-8}$	1.3262207×10^{-3}
1.0	$9.785338600 \times 10^{-8}$	2.0220546×10^{-3}

Table 6: Error for Problem 3

x	Two-steps with $v = \frac{2}{3}$
0.1	$5.99706000 \times 10^{-10}$
0.2	$2.007739000 \times 10^{-9}$
0.3	$5.35406900 \times 10^{-9}$
0.4	$4.544877000 \times 10^{-9}$
0.5	$8.650964000 \times 10^{-9}$
0.6	$1.513369800 \times 10^{-8}$
0.7	$1.7337878000 \times 10^{-8}$
0.8	$1.176467800 \times 10^{-8}$
0.9	$5.654679000 \times 10^{-9}$
1.0	$3.533287900 \times 10^{-8}$

4 Conclusion

Continuous hybrid scheme with off point was used with constructed orthogonal polynomials as basis function, developed through a collocation and interpolation technique. These method by analysis, were shown to be consistent and zero stable and hence convergent. Three selected problems have been considered to test the effectiveness and accuracy of the method. It is obvious from our table of results that the method is accurate and effective since the approximation closely estimate the analytic solution.

References

[1] Adeniyi, R. B., Alabi, M. O., Folaranmi, R. O. (2008). A Chebyshev Collocation Approach for a Continuous Formulation of Hybrid Methods for Initial Value Problems in Ordinary Differential Equations. Journal of Nigerian Association of Mathematical Physics, 12(1), 369-398.

[2] Adesanya, A. O., Anake, T. A., Oghoyon, G. J. (2009). Continuous Implicit Method for the solution of General Second Ordinary Differential Equations. Journal of Nigerian Association of Mathematical Physics, 15(1), 71-78.

[3] Adeyefa, E. O. (2014). A Collocation Approach for Continuous Hybrid Block Methods for Second Order Ordinary Differential Equations with Chebyshev Basis Function. Unpublished doctoral dissertation, University of Ilorin, Ilorin.

[4] Anake, T. A. (2011). Continuous Implicit Hybrid One-step methods for the Solution of Initial Value Problems of General Second Order Ordinary Differential Equations. Unpublished doctoral dissertation, Covenant University, Ota, Nigeria.

[5] Areo, E. A. (2013). A Self-Starting Linear Multistep Method for Direct Solution of Initial Value Problem of Second Order Ordinary Differential Equations. International Journal of Pure and Applied Mathematics, 85, 345-364.

[6] Awari, Y. S. (2013). Derivation and Application of Six Point Linear Multistep Numerical Methods for Solution of Second Order Initial Value Problems. ISOR Journal of Mathematics, 7, 23-29.

[7] Awoyemi, D. O., Kayode, S. J. (2008). A Minimal Order Collocation Method for Direct Solution of Initial Value Problems of General Second Order Ordinary Differential Equations. In Proceedings of the conference organized by National Mathematical Centre, Abuja, Nigeria.

[8] Awoyemi, D. O., Anake, T. A., Adesanya, A. O. (2012). A One Step Method for Solution of General Second Ordinary Differential Equations, 2, 159-163.

[9] Bamgbola, M. O., Adeniyi, R. B. (Accepted). Formulation of Discrete and Continuous Hybrid Method, Using Orthogonal Polynomials as Basis Function. Journal of Nigerian Association of Mathematical Physics.

[10] Bun, R. A., Vsilyev, Y. D. (1992). A Numerical Method for Solving Differential Equations of any orders. Computational Mathematics and Physics, 32, 317-330.

[11] Dahlquist, G. (1979). Some Properties of Linear Multistep and One Leg Method for Ordinary Differential Equations. Department of Computer Science, Royal Institute of Technology, Stockholm.

[12] Dahlquist, G. (1983). On One-leg Multistep Method. SIAM Journal on Numerical Analysis, 20, 1130-1146.

[13] Ekundayo, F., Adeniyi, R. B. (2014). An Orthogonal Collocation Approach for Continuous Formulation of Hybrid Methods for Initial Value Problems in Ordinary Differential Equations. Journal of Nigerian Association of Mathematical Physics (Accepted).

[14] Fatunla, S. O. (1988). Numerical Methods for Initial Value Problems in Ordinary Differential Equations. Academic Press Inc.

[15] Fatunla, S. O. (1991). Block Methods for Second Order Initial Value Problem (IVP). International Journal of Computer Mathematics, 41, 55-63.

[16] Haruna, Y., Adeniyi, R. B. (Accepted). A Collocation Technique Based on Orthogonal Polynomial for Construction of Continuous Hybrid Methods. Ilorin Journal of Science.

[17] Lambert, J. D. (1973). Computational Methods in Ordinary Differential System. John Wiley and Sons.

[18] Milne, W. E. (1953). Numerical Solution of Differential Equations. John Wiley and Sons.

[19] Mohammad, U., Yahaya, Y. A. (2010). Fully Implicit Four Point Block Backward Differentiation Formula for Solution of First Order Initial Value Problems. Leonardo Journal of Sciences, 16, 21-30.

[20] Onumanyi, P., Sirisena, W. V., Jator, S. N. (1999). Continuous Finite Difference Approximations for solving differential equations. International Journal of Computer and Mathematics, 72, 15-21.

[21] Owolabi, K. M. (2012). A Family of Implicit Higher Order Methods for the Numerical Integration of Second Order Differential Equations. Journal of Mathematical Theory and Modeling, 2, 67-76.

[22] Rosser, J. B. (1967). A Runge-Kutta for all Seasons. SIAM Review, 9, 417-452.

[23] Yusuf, Y., Onumanyi, P. (2005). New Multiple FDMs through Multistep collocation for y^n=f(x). In Proceedings of the seminar organized by the National Mathematical Center, Abuja, Nigeria.