The Physical Simulation of Oscillatory Differential Equations of Mass in Motion

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Abstract
This study delved into the practical application and simulation of oscillatory differential equations in the context of objects in motion. The methodology employed power series polynomials, ensuring that the fundamental properties of these functions were met. The new approach was applied to a range of oscillatory differential equations, including those related to harmonic motion, spring motion, dynamic mass motion, Betiss and Stiefel equations, and nonlinear differential equations. It has been shown to be computationally reliable, delivering improved accuracy and quicker convergence compared to the existing methods under consideration.

Keywords: Betiss and Stiefel, harmonic motion, physical application, mass in dynamic motion, spring of motion.

1 Introduction

Many physical problems remain unexplored and not yet fully addressed by researchers. While some problems in the fields of science, social science, and technology have been approached, many others remain uncharted territory. Oscillatory phenomena often play a key role in these areas, and one of the primary tools for modeling such oscillations is through the use of differential equations [1-3]. Researchers have employed oscillatory differential equations to tackle complex systems involving multiple variables [3]. This field of study is of great significance to numerical analysts as it enables the simulation of various phenomena in the realms of science, engineering, and social sciences [4-6]. For instance, it provides solutions for problems related to transportation, mass-spring systems, simple harmonic motion, and dynamic systems of objects, among others [6, 7]. These fields of study are simulated using oscillatory differential equations of the form.

$$\frac{d^2u}{dv^2} = f\left(v, u, \frac{du}{dv}\right), \; u(0) = \delta_0, \; \frac{du}{dv}(0) = \delta_1 \tag{1}$$

Hence, (1) continues to hold great importance for numerical analysts in the fields of science and technology, as it is used to numerically simulate various laws, theorems, and physical relationships [7-9].

In their work, the authors [10-15] attempted to simulate second-order oscillatory differential equation (1). However, the accuracy of their methods in terms of error was found to be notably low and not particularly encouraging.

The force governing the motion is consistently directed towards the equilibrium position and is directly proportional to the distance from it. In other words,

$$F = -kv \tag{2}$$

In this context, "F" represents the force, "v" denotes the displacement, and "k" is a constant—a relationship commonly referred to as Hooke's law. In simple terms, a spring-mass system involves a block attached to the free end of a spring. Typically, this system is employed to determine the period of an object undergoing simple harmonic motion [7, 8]. Moreover, it finds applications in a wide range of scenarios.

For example, a spring-mass system can be utilized to model (1). One of the most challenging aspects in the numerical solution of differential equations pertains to handling highly oscillatory systems. The primary goal of this research is to develop a method for simulating second-order oscillatory differential equations associated with objects in motion.

2 Methodology

The power series polynomial was used to develop the new method, for simulation of (1).
Let the approximate solution of

$$\gamma(\tau) = \sum_{j=0}^{\psi+\varsigma-1} \varpi_j \, \tau^j \qquad (3)$$

be the expected solution of (3) where $\tau \in [0, 1]$ and the number of interpolating and collocating points are ψ and ς. Differentiating (3) twice, yield

$$\gamma''(\tau) = \sum_{j=0}^{\psi+\varsigma-1} \tau(\tau-1)\varpi_j \, \tau^{j-2} \qquad (4)$$

Substituting (3) into (1) yield

$$\sum_{j=0}^{\psi+\varsigma-1} \tau(\tau-1)\varpi_j \, \tau^{j-2} = f(\tau, \gamma, \gamma') \qquad (5)$$

equation (3) is interpolated at $\psi = \frac{1}{6}, \frac{1}{4}$ while equation (5) is collocated at $\varsigma = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1$ which lead to

$$AX = U \qquad (6)$$

where

$$A = \begin{bmatrix}
1 & \frac{1}{6} & \frac{1}{36} & \frac{1}{216} & \frac{1}{1296} & \frac{1}{7776} & \frac{1}{46656} & \frac{1}{279936} & \frac{1}{1679616} & \frac{1}{10077696} & \frac{1}{60466176} \\
1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} & \frac{1}{4096} & \frac{1}{16384} & \frac{1}{65536} & \frac{1}{262144} & \frac{1}{1048576} \\
0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{3h^2} & \frac{5}{54h^2} & \frac{5}{216h^2} & \frac{7}{1296h^2} & \frac{7}{5832h^2} & \frac{1}{3888h^2} & \frac{5}{93312h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{3}{2h^2} & \frac{3}{4h^2} & \frac{5}{16h^2} & \frac{15}{128h^2} & \frac{21}{512h^2} & \frac{7}{512h^2} & \frac{9}{2048h^2} & \frac{45}{32768h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{4}{3h^2} & \frac{20}{27h^2} & \frac{10}{27h^2} & \frac{14}{81h^2} & \frac{56}{729h^2} & \frac{8}{243h^2} & \frac{10}{729h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{3}{h^2} & \frac{3}{h^2} & \frac{5}{2h^2} & \frac{15}{8h^2} & \frac{21}{16h^2} & \frac{7}{8h^2} & \frac{9}{16h^2} & \frac{45}{128h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{4}{h^2} & \frac{16}{3h^2} & \frac{160}{27h^2} & \frac{160}{27h^2} & \frac{448}{81h^2} & \frac{3584}{729h^2} & \frac{1024}{243h^2} & \frac{2560}{729h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{9}{2h^2} & \frac{27}{4h^2} & \frac{135}{16h^2} & \frac{1215}{128h^2} & \frac{5103}{512h^2} & \frac{5103}{512h^2} & \frac{19683}{2048h^2} & \frac{295245}{32768h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{5}{h^2} & \frac{25}{3h^2} & \frac{625}{54h^2} & \frac{3125}{216h^2} & \frac{21875}{1296h^2} & \frac{109375}{5832h^2} & \frac{78125}{3888h^2} & \frac{1953125}{93312h^2} \\
0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} & \frac{42}{h^2} & \frac{56}{h^2} & \frac{72}{h^2} & \frac{90}{h^2}
\end{bmatrix}$$

$$X = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_{n+\frac{1}{6}} & y_{n+\frac{1}{4}} & f_n & f_{n+\frac{1}{6}} & f_{n+\frac{1}{4}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{3}{4}} & f_{n+\frac{5}{6}} & f_{n+1} \end{bmatrix}^T$$

The unknown values of $a'_j s,\ j = 0(1)9$ are obtained by applying Gaussian elimination method and substituted into (3) to produce a continuous scheme with its derivatives of the form:

$$\gamma(\tau) = \sum_{j=\eta} \alpha_j^i(\tau) y_{n+j} + \sum_{j=0}^{1} \beta_j(\tau) f_{n+j} + \sum_{\varsigma \in \frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{5}{6}} \beta_\varsigma^i(\tau) f_{n+j},\ \eta = \frac{1}{6},\frac{1}{4},\ \varsigma = \frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{5}{6} \quad (7)$$

Where the values of $\alpha_j,\ j = \eta$ and $\beta_\varsigma,\ \varsigma \in \frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{5}{6}$ in equation (7) are

$$\alpha_{\frac{1}{6}} = 3 - 12\tau$$

$$\alpha_{\frac{1}{4}} = -2 + 12\tau$$

$$\beta_0 = \frac{109198219}{75246796800} - \frac{1562744609}{37623398400}\tau - \frac{1}{2}\tau^2 + \frac{601}{180}\tau^3 + \frac{1237}{90}\tau^4 - \frac{21959}{600}\tau^5 + \frac{28861}{450}\tau^6 - \frac{367}{5}\tau^7 + \frac{3693}{70}\tau^8 + \frac{108}{5}\tau^9 + \frac{95}{25}\tau^{10}$$

$$\beta_{\frac{1}{6}} = \frac{5336531}{154828800} - \frac{31769593}{7741400}\tau + \frac{162}{7}\tau^3 - \frac{11367}{70}\tau^4 + \frac{14013}{25}\tau^5 - \frac{28647}{25}\tau^6 + \frac{10152}{7}\tau^7 - \frac{5589}{5}\tau^8 + \frac{16848}{35}\tau^9 - \frac{15552}{175}\tau^{10}$$

$$\beta_{\frac{1}{4}} = -\frac{733019}{20995200} + \frac{41105527}{73483200}\tau - \frac{1024}{21}\tau^3 + \frac{123136}{315}\tau^4 - \frac{36864}{25}\tau^5 + \frac{721408}{225}\tau^6 - \frac{29696}{25}\tau^7 + \frac{16896}{5}\tau^8 - \frac{52224}{35}\tau^9 + \frac{49152}{175}\tau^{10}$$

$$\beta_{\frac{1}{3}} = \frac{25067281}{928972800} - \frac{40128343}{92897280}\tau + \frac{81}{2}\tau^3 - \frac{13797}{40}\tau^4 + \frac{55323}{40}\tau^5 - \frac{158139}{50}\tau^6 + \frac{30483}{7}\tau^7 - \frac{250533}{70}\tau^8 + 1620\tau^9 - \frac{7776}{25}\tau^{10}$$

$$\beta_{\frac{1}{2}} = -\frac{3043807}{250822656} + \frac{124090512}{627056640}\tau - 20\tau^3 + \frac{541}{3}\tau^4 - \frac{3866}{5}\tau^5 + \frac{28454}{15}\tau^6 - \frac{19512}{7}\tau^7 + \frac{16974}{7}\tau^8 - 1152\tau^9 + \frac{1152}{5}\tau^{10}$$

$$\beta_{\frac{2}{3}} = \frac{1213349}{103219200} - \frac{9986671}{51609600}\tau + \frac{81}{4}\tau^3 - \frac{1353}{20}\tau^4 + \frac{166617}{200}\tau^5 - \frac{106569}{50}\tau^6 + \frac{22977}{7}\tau^7 - \frac{209709}{70}\tau^8 + \frac{7452}{5}\tau^9 - \frac{7776}{25}\tau^{10}$$

$$\beta_{\frac{3}{4}} = -\frac{1367003}{146966400} + \frac{2257181}{14696640}\tau - \frac{1048}{3}\tau^3 + \frac{47872}{315}\tau^4 - \frac{2048}{3}\tau^5 + \frac{398848}{225}\tau^6 - \frac{19456}{7}\tau^7 + \frac{90624}{35}\tau^8 - \frac{92116}{7}\tau^9 + \frac{49152}{175}\tau^{10}$$

$$\beta_{\frac{5}{6}} = \frac{1216513}{464486400} - \frac{10067843}{232243200}\tau + \frac{162}{35}\tau^3 - \frac{3051}{70}\tau^4 + \frac{4941}{25}\tau^5 - \frac{12987}{25}\tau^6 + \frac{28944}{35}\tau^7 - \frac{27459}{35}\tau^8 + \frac{14256}{35}\tau^9 - \frac{15552}{175}\tau^{10}$$

$$\beta_1 = -\frac{6979241}{75246796800} + \frac{8281069}{5374771200}\tau - \frac{1}{6}\tau^3 + \frac{571}{360}\tau^4 - \frac{1459}{200}\tau^5 + \frac{8791}{450}\tau^6 - \frac{223}{7}\tau^7 + \frac{2181}{70}\tau^8 - \frac{84}{5}\tau^9 + \frac{96}{25}\tau^{10}$$

For $j = 0$, equation (7) is evaluated at the non-interpolating point $x_{n+\kappa},\ \kappa = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1$

For $j = \frac{1}{6}, \frac{1}{4}$ equation (7) are evaluated to produce the discreet schemes with its derivatives. The discreet scheme and its derivatives are combined in a block form as

$$\mathrm{A}\mathrm{Y}_m = Z\mathrm{N}_1 + h^2\left[\Omega \mathrm{N}_2 + B\mathrm{N}_3\right] \quad (8)$$

$$A=\begin{pmatrix}
1&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&2&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&6&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&7&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&8&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&10&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&\frac{12}{h}&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix},\ \mathrm{Y}_m=\begin{pmatrix} y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{4}}\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{2}{3}}\\ y_{n+\frac{3}{4}}\\ y_{n+\frac{5}{6}}\\ y_{n+1}\end{pmatrix},\ Z=\begin{pmatrix}
3&0\\ -1&0\\ -3&0\\ -5&0\\ -6&0\\ -7&0\\ -9&0\\ -\frac{12}{h}&-1\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0\\ -\frac{12}{h}&0
\end{pmatrix},\ N_1=\begin{pmatrix} y_n\\ y'_n\end{pmatrix},\ \Omega=\begin{pmatrix}
\frac{109198219}{75246796800h^2}\\
-\frac{695202}{225740390400h^2}\\
\frac{852491}{37623398400h^2}\\
-\frac{77509}{1672151040h^2}\\
\frac{571181}{10749542400h^2}\\
-\frac{478871}{8062156800h^2}\\
-\frac{826073}{5016453120h^2}\\
-\frac{1562744609}{37623398400h}\\
-\frac{126905078873}{4180377600h}\\
-\frac{1694300961851}{37623398400h}\\
-\frac{441985204589}{7524679680h}\\
-\frac{30120945190783}{37623398400h}\\
-\frac{358877107289}{4180377600h}\\
\frac{597039207199}{7524679680h}\\
\frac{2417108191009}{37623398400h}\\
\frac{8684951}{5374771200h}
\end{pmatrix},\ N_2=\left[f_n\right],\ N_3=\begin{pmatrix} f_{n+\frac{1}{6}}\\ f_{n+\frac{1}{4}}\\ f_{n+\frac{1}{3}}\\ f_{n+\frac{1}{2}}\\ f_{n+\frac{2}{3}}\\ f_{n+\frac{3}{4}}\\ f_{n+\frac{5}{6}}\\ f_{n+1}\end{pmatrix}$$

and

$$B=\begin{pmatrix}
\frac{109198219h^2}{75246796800} & \frac{5336531h^2}{154828800} & -\frac{733019h^2}{20995200} & \frac{25067281h^2}{928972800} & -\frac{1213349h^2}{103219200} & -\frac{1367003h^2}{146966400} & \frac{1216513h^2}{464486400} & \frac{6979241h^2}{75246796800} \\
-\frac{820373h^2}{1393459200} & \frac{2559607h^2}{440899200} & \frac{1467341h^2}{2786918400} & \frac{39581h^2}{752467968} & -\frac{174439h^2}{2786918400} & \frac{22417h^2}{440899200} & -\frac{20227h^2}{1393459200} & \frac{117499h^2}{22740390400} \\
\frac{173741}{77414400h^2} & \frac{1191037}{73483200h^2} & \frac{8930671}{464486400h^2} & \frac{632159}{125411328h^2} & -\frac{109861}{51609600h^2} & \frac{107227}{73483200h^2} & -\frac{86657h^2}{232243200} & \frac{439849h^2}{37623398400} \\
\frac{383057}{92897280h^2} & \frac{247577}{9797760h^2} & \frac{7878697}{185794560h^2} & \frac{2599903}{83607552h^2} & \frac{52613}{185794560h^2} & \frac{4981}{3265920h^2} & -\frac{47207}{92897280h^2} & \frac{94037}{5016453120h^2} \\
\frac{108341}{22118400h^2} & \frac{79919}{2624400h^2} & \frac{7036861}{132710400h^2} & \frac{1668539}{35831808h^2} & \frac{152069}{14745600h^2} & \frac{313}{328050h^2} & -\frac{22337}{66355200h^2} & \frac{167299}{10749542400h^2} \\
\frac{281603}{49766400h^2} & \frac{561937}{15746400h^2} & \frac{6328331}{99532800h^2} & \frac{1668539}{26873856h^2} & \frac{2077151}{99532800h^2} & \frac{97447}{15746400h^2} & \frac{21083}{22249766400h^2} & \frac{74989}{8062156800h^2} \\
\frac{20227}{206434h^2} & \frac{361087}{9797760h^2} & \frac{5973157}{61931520h^2} & \frac{6771913}{83607552h^2} & \frac{474569}{6881280h^2} & \frac{25553}{1399680h^2} & \frac{1127509}{30965760h^2} & \frac{1452743}{1003290624h^2} \\
-\frac{31769593h}{77414400} & \frac{41105527h}{73483200} & \frac{40128343h}{92897280} & \frac{124090513h}{627056640} & -\frac{9986671h}{51609600} & \frac{2257181h}{14696640} & -\frac{10067843h}{232243200} & \frac{8281069h}{5374771200} \\
\frac{21594670289h}{46448640} & -\frac{4280007077h}{3499200} & \frac{532876014413h}{464486400} & -\frac{134055047461h}{209018880} & \frac{320972388889h}{464486400} & -\frac{462338871h}{8164800} & \frac{5439073883h}{33177600} & -\frac{15177389611h}{2508226560} \\
\frac{7627652837h}{11059200} & -\frac{16665615493h}{91185400} & \frac{790472698507h}{464486400} & -\frac{596578867391h}{627056640} & \frac{52904100707h}{51609600} & -\frac{137780687h}{164025} & \frac{56479066153h}{232243200} & -\frac{337717095179h}{37623398400} \\
\frac{208927290389h}{2322243200} & -\frac{173897533321h}{73483200} & \frac{343686903407h}{154828800} & -\frac{778139715439h}{627056640} & \frac{88720288991h}{66355200} & -\frac{80511027823h}{73483200} & \frac{327411485h}{1032192} & -\frac{440495877253h}{37623398400} \\
\frac{94921775111h}{77414400} & -\frac{47404837813h}{14696640} & \frac{1405364732749h}{464486400} & -\frac{30300843589h}{17915904} & \frac{18810099997h}{10321920} & -\frac{109733517167h}{73483200} & \frac{100406334589h}{232243200} & -\frac{600381611141h}{37623398400} \\
\frac{305355437077h}{232243200} & -\frac{84720763523h}{24494400} & \frac{215281194251h}{66355200} & -\frac{379053845797h}{209018880} & \frac{907717913113h}{464486400} & -\frac{13074605239h}{8164800} & \frac{107668758653h}{232243200} & -\frac{214601277143}{12541132800} \\
\frac{94073939363h}{77414400} & -\frac{4194761041h}{1312200} & \frac{1392805460971h}{464486400} & -\frac{1051003431839h}{627056640} & \frac{93219247043h}{51609600} & -\frac{6797094653h}{4592700} & \frac{568634503h}{1327104} & -\frac{595026871403h}{37623398400} \\
\frac{32644966403h}{33177600} & -\frac{190202395273h}{73483200} & \frac{75184098019h}{30965760} & -\frac{850970104687h}{627056640} & \frac{679318145561h}{464486400} & -\frac{2515778117h}{2099520} & \frac{89534648373h}{25804800} & -\frac{481798339717h}{37623398400} \\
\frac{812443h}{15482880} & -\frac{6670217h}{73483200} & \frac{148718413h}{464486400} & -\frac{7292783h}{627056640} & \frac{28831121h}{51609600} & -\frac{36489839h}{73483200} & \frac{97427581h}{232243200} & -\frac{311983487h}{7524679680}
\end{pmatrix}$$

equation (8) is multiplied by the inverse of A to have a hybrid block method of the form

$$A^{(0)}\mathrm{Y}_m = A^{-1}ZN_1 + h^2\left[A^{-1}\Omega N_2 + A^{-1}BN_3\right] \qquad (9)$$

Equation (9) can be written as follows

$$y_{n+\frac{1}{6}} = y_n + \frac{hy'_n}{6} + h^2\left[\begin{array}{l} \frac{9649609}{1763596800} f_n + \frac{4925}{145152} f_{n+\frac{1}{6}} - \frac{200876}{3444525} f_{n+\frac{1}{4}} + \frac{979999}{21772800} f_{n+\frac{1}{3}} - \frac{612761}{29393280} f_{n+\frac{3}{4}} \\ + \frac{49583}{2419200} f_{n+\frac{2}{3}} - \frac{56132}{3444525} f_{n+\frac{3}{4}} + \frac{50143}{10886400} f_{n+\frac{5}{6}} - \frac{57859}{352719360} f_{n+1} \end{array}\right]$$

$$y_{n+\frac{1}{4}} = y_n + \frac{hy'_n}{4} + h^2\left[\begin{array}{l} \frac{1844099}{206438400} f_n + \frac{781353}{11468800} f_{n+\frac{1}{6}} - \frac{4701}{44800} f_{n+\frac{1}{4}} + \frac{1858113}{22937600} f_{n+\frac{1}{3}} - \frac{128467}{3440640} f_{n+\frac{3}{4}} \\ + \frac{839997}{22937600} f_{n+\frac{2}{3}} - \frac{11731}{403200} f_{n+\frac{3}{4}} + \frac{94257}{11468800} f_{n+\frac{5}{6}} - \frac{20123}{68812800} f_{n+1} \end{array}\right]$$

$$y_{n+\frac{1}{3}} = y_n + \frac{hy'_n}{3} + h^2\left[\begin{array}{l} \frac{68291}{5511240} f_n + \frac{8753}{85050} f_{n+\frac{1}{6}} - \frac{502016}{3444525} f_{n+\frac{1}{4}} + \frac{238}{2025} f_{n+\frac{1}{3}} - \frac{12349}{229635} f_{n+\frac{3}{4}} \\ + \frac{17923}{340200} f_{n+\frac{2}{3}} - \frac{144128}{3444525} f_{n+\frac{3}{4}} + \frac{67}{5670} f_{n+\frac{5}{6}} - \frac{5791}{13778100} f_{n+1} \end{array}\right]$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2hy'_n}{3} + h^2\left[\begin{array}{l} \frac{90224}{3444525} f_n + \frac{3448}{14175} f_{n+\frac{1}{6}} - \frac{1077248}{3444525} f_{n+\frac{1}{4}} + \frac{12902}{42525} f_{n+\frac{1}{3}} - \frac{20368}{229635} f_{n+\frac{3}{4}} \\ + \frac{238}{2025} f_{n+\frac{2}{3}} - \frac{45056}{492075} f_{n+\frac{3}{4}} + \frac{1096}{42525} f_{n+\frac{5}{6}} - \frac{3154}{3444525} f_{n+1} \end{array}\right]$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3hy'_n}{4} + h^2\left[\begin{array}{l} \frac{136011}{4587520} f_n + \frac{3190833}{11468800} f_{n+\frac{1}{6}} - \frac{15867}{15867} f_{n+\frac{1}{4}} + \frac{8028477}{22937600} f_{n+\frac{1}{3}} - \frac{102897}{1146880} f_{n+\frac{3}{4}} \\ + \frac{3295809}{22937600} f_{n+\frac{2}{3}} - \frac{4701}{44800} f_{n+\frac{3}{4}} + \frac{67797}{2293760} f_{n+\frac{5}{6}} - \frac{24021}{22937600} f_{n+1} \end{array}\right]$$

$$y_{n+\frac{5}{6}} = y_n + \frac{5hy'_n}{6} + h^2\left[\begin{array}{l} \frac{2335225}{70543872} f_n + \frac{136375}{435456} f_{n+\frac{1}{6}} - \frac{54500}{137781} f_{n+\frac{1}{4}} + \frac{38375}{96768} f_{n+\frac{1}{3}} - \frac{533125}{5878656} f_{n+\frac{3}{4}} \\ + \frac{148375}{870912} f_{n+\frac{2}{3}} - \frac{15500}{137781} f_{n+\frac{3}{4}} + \frac{4925}{145152} f_{n+\frac{5}{6}} - \frac{83375}{70543872} f_{n+1} \end{array}\right]$$

$$y_{n+1} = y_n + hy'_n + h^2\left[\frac{503}{12600} f_n + \frac{27}{70} f_{n+\frac{1}{6}} - \frac{256}{525} f_{n+\frac{1}{4}} + \frac{351}{700} f_{n+\frac{1}{3}} - \frac{11}{105} f_{n+\frac{3}{4}} + \frac{351}{1400} f_{n+\frac{2}{3}} - \frac{256}{1575} f_{n+\frac{3}{4}} + \frac{27}{350} f_{n+\frac{5}{6}}\right]$$

$$y'_{n+\frac{1}{6}} = y_n + h\left[\begin{array}{l} \frac{6117617}{146966400} f_n + \frac{1571}{4050} f_{n+\frac{1}{6}} - \frac{673996}{1148175} f_{n+\frac{1}{4}} + \frac{802813}{1814400} f_{n+\frac{1}{3}} - \frac{15413}{76545} f_{n+\frac{3}{4}} + \frac{356563}{1814400} f_{n+\frac{2}{3}} \\ - \frac{178996}{1148175} f_{n+\frac{3}{4}} + \frac{1247}{28350} f_{n+\frac{5}{6}} - \frac{229633}{146966400} f_{n+1} \end{array}\right]$$

$$y'_{n+\frac{1}{4}} = y_n + h\left[\begin{array}{l} \frac{1070131}{25804800} f_n + \frac{300429}{716800} f_{n+\frac{1}{6}} - \frac{52279}{100800} f_{n+\frac{1}{4}} + \frac{1211031}{1814400} f_{n+\frac{1}{3}} - \frac{10481}{53760} f_{n+\frac{3}{4}} + \frac{547641}{2867200} f_{n+\frac{2}{3}} \\ - \frac{15289}{100800} f_{n+\frac{3}{4}} + \frac{30699}{716800} f_{n+\frac{5}{6}} - \frac{39299}{25804800} f_{n+1} \end{array}\right]$$

$$y'_{n+\frac{1}{3}} = y_n + h\left[\begin{array}{l} \frac{381439}{9185400} f_n + \frac{5897}{14175} f_{n+\frac{1}{6}} - \frac{545792}{1148175} f_{n+\frac{1}{4}} + \frac{53141}{113400} f_{n+\frac{1}{3}} - \frac{15286}{76545} f_{n+\frac{3}{4}} + \frac{22061}{113400} f_{n+\frac{2}{3}} \\ - \frac{177152}{1148175} f_{n+\frac{3}{4}} + \frac{617}{14175} f_{n+\frac{5}{6}} - \frac{14201}{9185400} f_{n+1} \end{array}\right]$$

$$y'_{n+\frac{1}{2}} = y_n + h\left[\begin{array}{l} \frac{8329}{201600} f_n + \frac{297}{700} f_{n+\frac{1}{6}} - \frac{116}{225} f_{n+\frac{1}{4}} + \frac{13149}{22400} f_{n+\frac{1}{3}} - \frac{11}{105} f_{n+\frac{3}{4}} + \frac{3699}{22400} f_{n+\frac{2}{3}} - \frac{212}{1575} f_{n+\frac{3}{4}} \\ + \frac{21}{700} f_{n+\frac{5}{6}} - \frac{281}{201600} f_{n+1} \end{array}\right]$$

$$y'_{n+\frac{2}{3}} = y_n + h\left[\begin{array}{l} \frac{47611}{1148175} f_n + \frac{5944}{14175} f_{n+\frac{1}{6}} - \frac{569344}{1148175} f_{n+\frac{1}{4}} + \frac{7904}{14175} f_{n+\frac{1}{3}} - \frac{752}{76545} f_{n+\frac{3}{4}} + \frac{4019}{14175} f_{n+\frac{2}{3}} \\ - \frac{28672}{164025} f_{n+\frac{3}{4}} + \frac{664}{14175} f_{n+\frac{5}{6}} - \frac{1844}{1148175} f_{n+1} \end{array}\right]$$

$$y'_{n+\frac{3}{4}} = y_n + h\left[\begin{array}{l} \frac{118827}{2867200} f_n + \frac{43011}{102400} f_{n+\frac{1}{6}} - \frac{5583}{11200} f_{n+\frac{1}{4}} + \frac{1608903}{2867200} f_{n+\frac{1}{3}} - \frac{261}{17920} f_{n+\frac{3}{4}} + \frac{945513}{2867200} f_{n+\frac{2}{3}} \\ - \frac{1473}{11200} f_{n+\frac{3}{4}} + \frac{31347}{716800} f_{n+\frac{5}{6}} - \frac{4443}{2867200} f_{n+1} \end{array}\right]$$

$$y'_{n+\frac{5}{6}} = y_n + h\left[\begin{array}{l} \frac{243865}{5878656} f_n + \frac{475}{1134} f_{n+\frac{1}{6}} - \frac{22700}{45927} f_{n+\frac{1}{4}} + \frac{40325}{72576} f_{n+\frac{1}{3}} - \frac{125}{15309} f_{n+\frac{3}{4}} + \frac{22475}{72576} f_{n+\frac{2}{3}} \\ - \frac{2900}{45927} f_{n+\frac{3}{4}} + \frac{85}{1134} f_{n+\frac{5}{6}} - \frac{10025}{5878656} f_{n+1} \end{array}\right]$$

$$y'_{n+1} = y_n + h\left[\begin{array}{l} \frac{503}{12600} f_n + \frac{81}{175} f_{n+\frac{1}{6}} - \frac{1024}{1575} f_{n+\frac{1}{4}} + \frac{1053}{1400} f_{n+\frac{1}{3}} - \frac{22}{105} f_{n+\frac{3}{4}} + \frac{1053}{1400} f_{n+\frac{2}{3}} - \frac{1024}{1575} f_{n+\frac{3}{4}} \\ + \frac{81}{175} f_{n+\frac{5}{6}} - \frac{503}{12600} f_{n+1} \end{array}\right]$$

3 Basic Properties of the new Method

We will scrutinize the assessment of the novel approach, encompassing various properties such as order, error constant, consistency, convergence, zero-stability, and stability region [16, 17], among others.

3.1 Order and Error constant of the Method

In determining the order and error constant of the new method (9), we define the linear difference operator L associated with equation (9) as

$$L[y(x);h] = Y_m - A^{-1}ZN_1 - h^2\left[A^{-1}\Omega N_2 + A^{-1}BN_3\right] \tag{10}$$

Corollary 1 [17]

Compare the linear operator (10) with the truncation error $C_{09}h^{09}y^{09}(x_n)+0(h^{10})$.

Proof

The linear difference operators (10) is compared with the new method (9) as

$$\left.\begin{aligned}
l_{\frac{1}{6}}[y(x_n);h] &= y\left(x_n+\frac{1}{6}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{1}{4}}[y(x_n);h] &= y\left(x_n+\frac{1}{4}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{1}{3}}[y(x_n);h] &= y\left(x_n+\frac{1}{3}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{1}{2}}[y(x_n);h] &= y\left(x_n+\frac{1}{2}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{2}{3}}[y(x_n);h] &= y\left(x_n+\frac{2}{3}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{3}{4}}[y(x_n);h] &= y\left(x_n+\frac{3}{4}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_{\frac{5}{6}}[y(x_n);h] &= y\left(x_n+\frac{5}{6}h\right)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)\\
l_1[y(x_n);h] &= y(x_n+h)-\left(\alpha_{\frac{1}{6}}\left(x_n+\frac{1}{6}h\right)+\alpha_{\frac{1}{4}}\left(x_n+\frac{1}{4}h\right)+h^2\sum_{j=0}^{1}\left(\beta_\varsigma(x)f_{n+\varsigma}+\beta_\varsigma(x)f_{n+\varsigma}\right)\right)
\end{aligned}\right\} \quad (11)$$

Corollary 2 [17]

The local truncation error of (9) is assume $y(x)$ to be sufficiently differentiable and expanding $y(x_n+qh)$ and $y(x_n+jh)$ about x_n using Taylor series to have

$l_{\frac{1}{6}}[y(x_n);h]=(1.2415\times10^{-12}), l_{\frac{1}{4}}[y(x_n);h]=(2.2041\times10^{-12}), l_{\frac{1}{3}}[y(x_n);h]=(3.1629\times10^{-12}),$

$l_{\frac{1}{2}}[y(x_n);h]=(5.0339\times10^{-12}), l_{\frac{2}{3}}[y(x_n);h]=(6.9050\times10^{-12}), l_{\frac{3}{4}}[y(x_n);h]=(7.8638\times10^{-12}),$

$l_{\frac{5}{6}}[y(x_n);h]=(8.8264\times10^{-12}), l_1[y(x_n);h]=(1.0068\times10^{-11})$

Proof

Expanding the term Υ_m and N_3 using a Taylor series about x_n respectively and then collecting their like elements to the power of h gives

$$l_{\frac{1}{6}}[y(x_n);h] = (1.2415\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_{\frac{1}{4}}[y(x_n);h] = (2.2041\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_{\frac{1}{3}}[y(x_n);h] = (3.1629\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_{\frac{1}{2}}[y(x_n);h] = (5.0339\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_{\frac{2}{3}}[y(x_n);h] = (6.9050\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_{\frac{3}{4}}[y(x_n);h] = (7.8638\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_{\frac{5}{6}}[y(x_n);h] = (8.8264\times10^{-12})h^9 y^{(9)}(x_n) + 0(h^{10})$$
$$l_1[y(x_n);h] = (1.0068\times10^{-11})h^9 y^{(9)}(x_n) + 0(h^{10})$$

Hence, from the above results, the order of the new method (9) is 9, and the error constants is $C = \begin{pmatrix} 1.2415\times10^{-12}, 2.2041\times10^{-12}, 3.1629\times10^{-12}, 5.0339\times10^{-12}, \\ 6.9050\times10^{-12}, 7.8638\times10^{-12}, 8.8264\times10^{-12}, 1.0068\times10^{-11} \end{pmatrix}^T$.

3.2 Consistency

Definition 1 [17]

The new method (9) is consistent because it is of order 9.

3.3 Zero-stability of the Method

For zero stability, we consider the characteristic function of the equation below:

$$\left|\left[\lambda B^{(0)} - B^i\right]\right| = \left|\lambda\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}\right| = 0$$

$\lambda^8 - \lambda^7 = 0, 0, 0, 0, 0, 0, 0, 0, 1$

Since the roots of the equations lies between 0 and 1, hence the new method is zero stable see [16].

3.3 Convergent

Theorem 1 [17]

According to Dalquist theorem, the new method is convergent since it is consistence and zero-stable see [16].

3.5 Linear Stability

Definition 3 [18]

The stability region of a new method is the set of complex values λh for which all solutions of the test problem $y'' = -\lambda^2 y$ remain bounded as $n \to \infty$.

The concept of A-stability according to [18] is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)} y \tag{12}$$

To yield

$$Y_m = \mu(z) Y_{m-1}, \; z = \lambda h \tag{13}$$

Where $\mu(z)$ is the amplification matrix of the form

where [illegible] is the amplification matrix given by

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^2\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^2\eta^{(1)}\right) \tag{14}$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of new method (9) is given as

$$\zeta = -\frac{\begin{pmatrix} 24799949\,719675695z^8 - 1167073163\,739266043z^7 + 27\,128030061\,143833235z^6 - \\ 515\,556735008\,654413944z^5 + 6539\,326196102\,856181344z^4 - 65866\,416469167\,064393152z^3 \\ + 430104\,648937877\,518309632z^2 - 1874456\,030584895\,333990400z + 3669028\,117771997\,675520000 \end{pmatrix}}{\begin{matrix} 29255954\,595840000z^8 - 1172188580\,806656000z^7 + 28\,951692668\,043264000z^6 - \\ 513\,945205576\,040448000z^5 + 6754\,848844724\,305920000z^4 - 64936\,984916199\,997440000z^3 \\ + 435626\,312980066\,467840000z^2 - 1834514\,058885998\,837760000z + 3669028\,117771997\,675520000 \end{matrix}}$$

4 Mathematical Illustration

The new method was employed for simulating various types of oscillatory differential equations. Firstly, we conducted a numerical simulation of oscillatory differential equation (1) in a motion to identify the characteristics of mass in a spring, dynamic mass, and equilibrium in harmonic form. Secondly, we simulated oscillatory differential equation (1) with an external force "F" to examine its impact on the system's behavior. Lastly, we conducted oscillatory simulations of (1) in both linear and nonlinear forms.

The notations below are used in the results

ES: Exact Solution
CS: Computed Solution
NM: New Method
ENM: Error in New Method
E[19]: Error in [19]
E[12]: Error in [12]
E[13]: Error in [13]
E[20]: Error in [20]
E[21]: Error in [21]
E[22]: Error in [22]
E[23]: Error in [23]

Example 1

Consider the mechanical oscillatory differential equation in harmonic motion, of an object which stretches a spring 6 inches in equilibrium.

i. Set up the equation of motion and find its general solution.
ii. Find the displacement of the object for $t > 0$, if it's initially displaced 18 inches above equilibrium and given a downward velocity of $3\frac{ft}{s}$.

From Newton's second law of motion, we have

$$mu''+cu'+ku = F \tag{15}$$

By setting $c = 0$ and $F = 0$, we get

$$mu''+ku = 0 \Rightarrow u''+\frac{k}{m}u = 0 \tag{16}$$

The equation of the weight of the object is given as follow:

$$mg = k\Delta l \Rightarrow \frac{k}{m} = \frac{g}{\Delta l} \tag{17}$$

Substituting $g = 32\frac{ft}{s^2}, \Delta l = \frac{6}{12}ft$ into (17) we obtain

$$\frac{k}{m} = \frac{32}{\frac{6}{12}} = 64 \tag{17}$$

Substituting equation (18) into the equation (16) we get

$$u''+64u = 0 \tag{18}$$

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus, $u(0) = \frac{3}{2}, u'(0) = -3$ and $h = 0.1$. We make use of (18) as

$$dsolver\left(\left\{u''(v) + 64u(v) = 0, u(0) = \frac{3}{2}, u'(0) = -3\right\}\right) \tag{20}$$

We obtain the exact solution (20) as

$$u(v) = -\frac{3}{8}\sin(8v) + \frac{3}{2}\cos(8v) \tag{21}$$

Source: [19].

Example 2

The second order mechanical oscillatory differential equation in a spring of motion is consider.

A $128lb$ weight is attached to a spring having a spring constant of $64lb/ft$. The weight is started in motion with no initial velocity by displacing it $6inches$ above the equilibrium position and by simultaneously applying to the weight an external force $F_4(v) = 8\sin 4v$. Assuming no air resistance, compute the subsequent motion of the weight at $t: 0.01 \le v \le 0.10$.

Now, we model this problem into a mathematical model and then apply our method to compute the motion on the weight attached to the spring. Here,

$$m = 4, k = 64, b = 0, and\ F_4(v) = 8\sin 4v$$

Thus, problem 3 boils down to

$$\frac{d^2u}{dv^2} + 16u = 2\sin 4v, u(0) = -\frac{1}{2}, u'(0) = 0 \tag{22}$$

with the exact solution of (22) is given by,

$$u(v) = -\frac{1}{2}\cos 4v + \frac{1}{16}\sin 4v - \frac{1}{4}v\cos 4v \tag{23}$$

Source: [12, 13].

Example 3

Consider the mass in a dynamic motion that is coined into linear oscillatory form of differential equation (1).

A mass of $10\,kg$ is attached to a spring having a constant spring of $140\ N/M$. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force $F(v)=5\sin v$. Find the subsequent motion of the mass $(v: 0.10 \le v \le 1.00)$ if the force due to air resistance is $90\left(\frac{du}{dv}\right)N$.

We apply the same procedure, where $m=10,\ k=140,\ a=90\ and\ F(v)=5\sin v$ example 3 reduces to

$$dsolver\left(\left\{\frac{d^2u}{dv^2}+9\frac{du}{dv}+14y(u)=\frac{1}{2}\sin(v),\ u(0)=0,\ u'(0)=-1\right\}\right) \quad (24)$$

with the exact solution of (24) is given by,

$$u(v)=\frac{1}{500}\left(-90\exp(-2v)+99\exp(-7v)+13\sin v-9\cos v\right) \quad (25)$$

Source [12, 13, 20].

Example 4

Consider the linear oscillatory differential equation in Betiss and Stiefel form

$$\frac{d^2u_1}{dv^2}+\frac{du_1}{dv}=0.001\cos(v),\ u_1(0)=1,\ \frac{du_1}{dv}=0 \quad (26)$$

$$\frac{d^2u_2}{dv^2}+\frac{du_2}{dv}=0.001\sin(v),\ u_1(0)=0,\ \frac{du_1}{dv}=0.9995 \quad (27)$$

With exact solution of (26) and (27) as

$$u_1(v)=\cos(v)+0.0005v\sin(v) \quad (28)$$

$$u_2(v)=\sin(v)-0.0005v\cos(v) \quad (29)$$

Source [21, 22]

Example 5:

Consider the nonlinear oscillatory differential equation

$$\frac{d^2u}{dv^2}-4yu'+8u=v^3,\ u(0)=2, u'(0)=4, \quad (30)$$

Whose exact solution is

$$y(v)=\exp(2v)\left(2\cos(2v)-\frac{3}{64}\sin(2v)\right)+\frac{3v}{32}+\frac{3v^2}{16}+\frac{v^2}{8} \quad (31)$$

Source: [22, 23].

5 Results and Discussion

Table 1: Computation of NM with [19] when solving example 1

v	**ES**	**CS**	**ENM**	**E[19]**
0.1	0.77605152993342709579	0.77605152993274408426	6.8301(-13)	3.3496(-07)

0.2	-0.41863938459249752594	-0.41863938459387367324	1.3762(-12)	1.6371(-06)
0.3	-1.3593892660185498469	-1.3593892660195541960	1.0056(-12)	3.2716(-06)
0.4	-1.4755518599067871611	-1.4755518599060687296 0	7.1843(-13)	3.5979(-06)
0.5	-0.69666449555494477770	-0.69666449555213113975	2.8136(-12)	1.3589(-06)
0.6	0.50481020347261010590	0.50481020347619324768	3.5831(-12)	2.9143(-06)
0.7	1.4000738069674951883	1.40007380696939826270	1.9031(-12)	6.7226(-06)
0.8	1.4460714263183540043	1.44607142631665691830	1.6971(-12)	7.0589(-06)
0.9	0.61490152285494961183	0.61490152284989092499	5.0587(-12)	2.6543(-06)
1.0	-0.58925939319668845548	0.58925939320237650700	5.6881(-12)	4.6056(-06)

See [22, 23].

Table 2: Computation of NM with [12, 13] when solving example 2

v	**ES**	**CS**	**ENM**	**E[12]**	**E[13]**
0.1	-0.49959872021047678004	-0.49959872021047678004	0.0000(00)	1.6621(-09)	1.0000(-19)
0.2	-0.49839019330974949646	-0.49839019330974949646	0.0000(00)	1.1586(-08)	4.1000(-19)
0.3	-0.49636836974027966301	-0.49636836974027966301	0.0000(00)	2.9743(-08)	9.1000(-19)
0.4	-0.49352852660817937130	-0.49352852660817937130	0.0000(00)	5.6076(-08)	1.6600(-18)
0.5	-0.48986728796894500998	-0.48986728796894500998	0.0000(00)	9.0504(-08)	2.6200(-18)
0.6	-0.48538264289709933476	-0.48538264289709933476	0.0000(00)	1.3291(-07)	3.8000(-18)
0.7	-0.48007396129056685722	-0.48007396129056685722	0.0000(00)	1.8317(-07)	5.2000(-18)
0.8	-0.47394200736436189072	-0.47394200736436189072	0.0000(00)	2.4110(-07)	6.8500(-18)
0.9	-0.46698895079202783994	-0.46698895079202783994	0.0000(00)	3.0653(-07)	8.7500(-18)
1.0	-0.45921837545722401274	-0.45921837545722401274	0.0000(00)	3.7922(-07)	1.0850(-17)

See [12, 13].

Table 3: Computation of NM with [12, 13, 20] when solving example 3

V	**ES**	**CS**	**ENM**	**E[12]**	**E[13]**	**E[20]**
0.1	-0.06436205154552458248	-0.06436205154550692713	1.7655(-14)	1.2744(-08)	2.0453(-10)	4.4268(-09)
0.2	-0.08430720522644774945	-0.08430720522643379455	1.3955(-14)	3.0442(-08)	4.8485(-10)	2.2383(-08)
0.3	-0.08405225313390041905	-0.08405225313389384414	6.5749(-15)	4.1501(-08)	6.6174(-10)	3.5865(-08)
0.4	-0.07529304213333374810	-0.07529304213333305897	6.8913(-16)	4.5385(-08)	7.2649(-10)	4.2157(-08)
0.5	-0.06357063960355798563	-0.06357063960356088722	2.9016(-15)	4.4298(-08)	7.1295(-10)	4.2895(-08)
0.6	-0.05142117069384508163	-0.05142117069384974188	4.6603(-15)	4.0466(-08)	6.5550(-10)	4.0288(-08)
0.7	-0.03993052956438697070	-0.03993052956439220056	5.2299(-15)	3.5475(-08)	5.7884(-10)	3.6051(-08)
0.8	-0.02949865862803573900	-0.02949865862804086216	5.1232(-15)	3.0285(-08)	4.9808(-10)	3.1287(-08)
0.9	-0.02021269131259124546	-0.02021269131259391333	2.6679(-15)	2.5408(-08)	4.2140(-10)	2.6618(-08)
1.0	-0.01202699425403169607	-0.01202699425403402038	2.3243(-15)	2.1071(-08)	3.5257(-10)	2.2352(-08)

See [12, 13, 20].

Table 4: Computation of NM with [21, 22] when solving (26)

V	**ES**	**CS**	**ENM**	**E[21]**	**E[22]**
0.1	0.09978366643856425102	0.09978366643856425102	0.0000(00)	1.2567(-12)	1.0170(-12)
0.2	0.19857132413727709130	0.19857132413727709130	0.0000(00)	2.1140(-12)	1.4285(-11)
0.3	0.29537690618797073421	0.29537690618797073421	0.0000(00)	2.3764(-12)	4.9557(-11)
0.4	0.38923413010984991465	0.38923413010984991465	0.0000(00)	3.4242(-12)	1.0161(-10)
0.5	0.47920614296373040709	0.47920614296373040709	0.0000(00)	3.3944(-12)	1.7416(-10)
0.6	0.56439487271056245371	0.56439487271056245371	0.0000(00)	3.3436(-12)	2.6425(-10)
0.7	0.64394999247214148272	0.64394999247214148272	0.0000(00)	4.2949(-12)	3.7579(-10)
0.8	0.71707740821578389546	0.71707740821578389546	0.0000(00)	4.2574(-12)	5.0602(-10)
0.9	0.78304718514176158945	0.78304718514176158945	0.0000(00)	5.2344(-12)	6.5904(-10)
1.0	0.84120083365496243679	0.84120083365496243679	0.0000(00)	6.2265(-12)	8.3225(-10)

See [21, 22].

Table 5: Computation of NM with [21, 22] when solving (27)

V	**ES**	**CS**	**ENM**	**E[21]**	**E[22]**
0.1	0.99500915694885810751	0.99500915694885810750	0.0000(00)	2.8269(-12)	1.0169(-11)
0.2	0.98008644477432113724	0.98008644477432113723	0.0000(00)	5.8994(-12)	2.0390(-11)
0.3	0.95538081715660522058	0.95538081715660522057	0.0000(00)	6.8309(-12)	1.5451(-13)
0.4	0.92113887767134681290	0.92113887767134681288	0.0000(00)	1.4991(-12)	8.1063(-11)
0.5	0.87770241827502376687	0.87770241827502376685	0.0000(00)	1.8395(-12)	2.5377(-10)
0.6	0.82550500765169680785	0.82550500765169680783	0.0000(00)	1.6559(-11)	5.4848(-10)
0.7	0.76506766347502161813	0.76506766347502161811	0.0000(00)	1.2970(-11)	9.9571(-10)
0.8	0.69699365178352523002	0.69699365178352523001	0.0000(00)	8.4312(-11)	1.6260(-10)
0.9	0.62196246537999682400	0.62196246537999682400	0.0000(00)	5.3240(-11)	2.4697(-10)
1.0	0.54072304136054366565	0.54072304136054366565	0.0000(00)	3.2126(-11)	3.5575(-10)

See [21, 22].

Table 6: Computation of NM with [22, 23] when solving example 5

V	**ES**	**CS**	**ENM**	**E[22]**	**E[23]**
0.1	2.3941125769963956181	2.39411257699639563790	1.9800(-17)	7.1426(-08)	5.1070(-06)
0.2	2.7481413324264235256	2.74814133242642358080	5.5200(-17)	1.7491(-07)	1.4959(-05)
0.3	3.0078669405110678859	3.00786694051106799770	1.1180(-16)	3.6449(-07)	2.7853(-05)
0.4	3.1017624057742078185	3.10176240577420801430	1.9580(-16)	6.1898(-07)	4.2891(-05)
0.5	2.9395431007452620774	2.93954310074526238920	3.1180(-16)	6.9889(-07)	6.7031(-05)
0.6	2.4118365344157147255	2.41183653441571519130	4.6580(-16)	1.4794(-06)	1.0264(-04)
0.7	1.3915548304898433104	1.39155483048984396930	6.5890(-16)	2.1022(-06)	1.4491(-04)
0.8	-0.262326758334357631	-0.26232675833435674263	8.8837(-16)	2.8409(-06)	1.9091(-04)
0.9	-2.697771160773070925	-2.69777116077306977980	1.1452(-15)	3.6689(-06)	2.3973(-04)
1.0	-6.058560720845666951	-6.05856072084566553990	1.4111(-15)	4.5617(-06)	2.9467(-04)

See [22, 23]

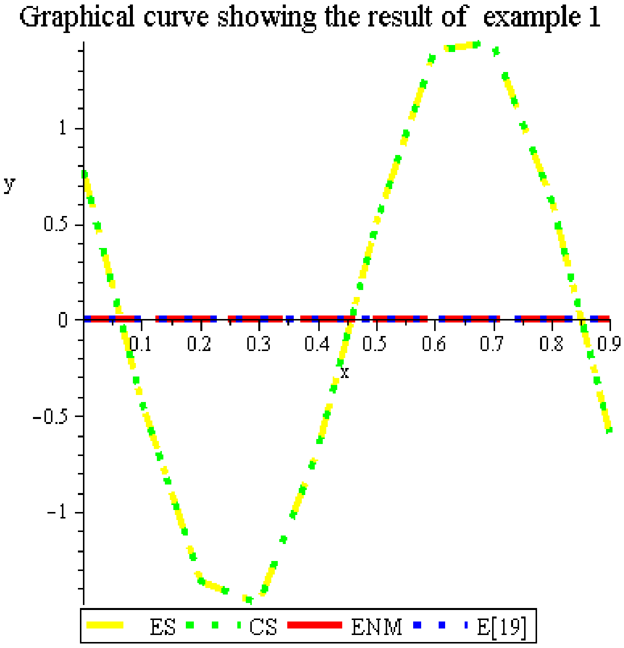

Figure 1: Textual graph of table 1

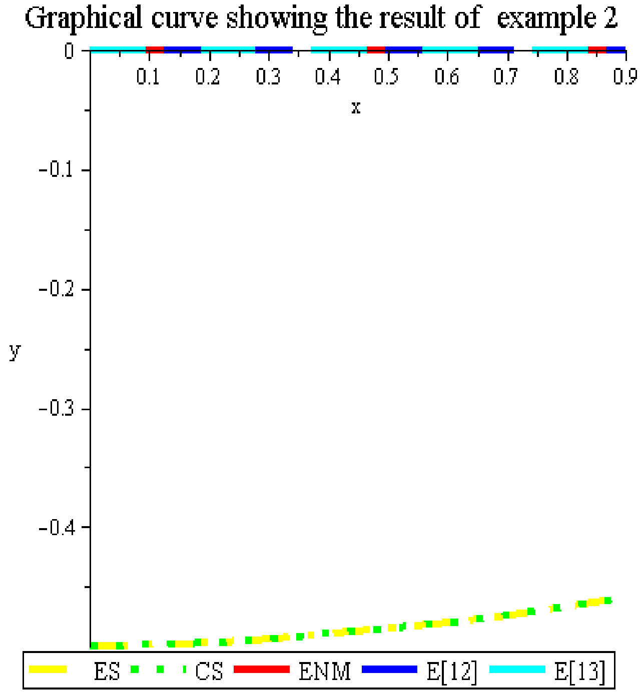

Figure 2: Textual graph of table 2

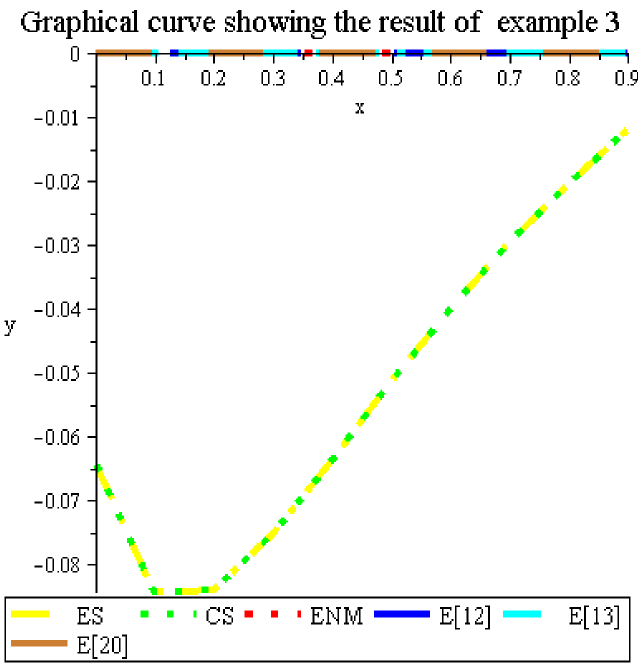

Figure 3: Textual graph of table 3

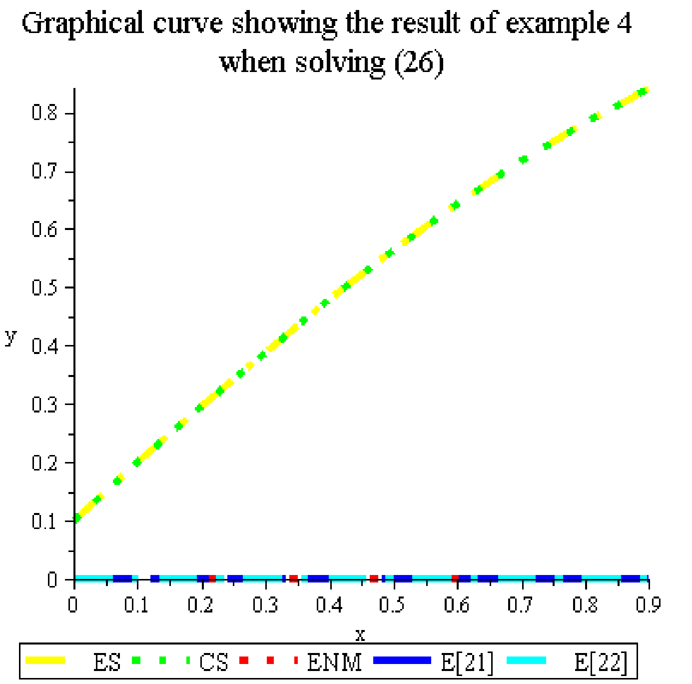

Figure 4: Textual graph of table 4

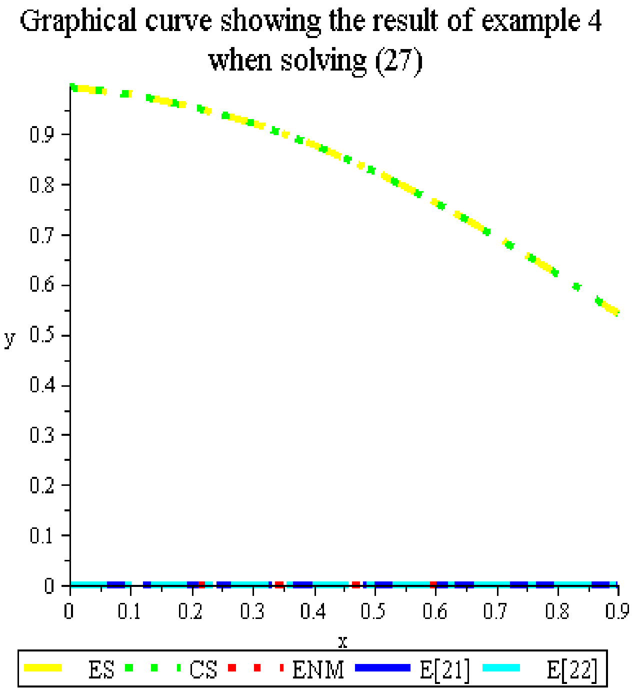

Figure 4: Textual graph of table 5

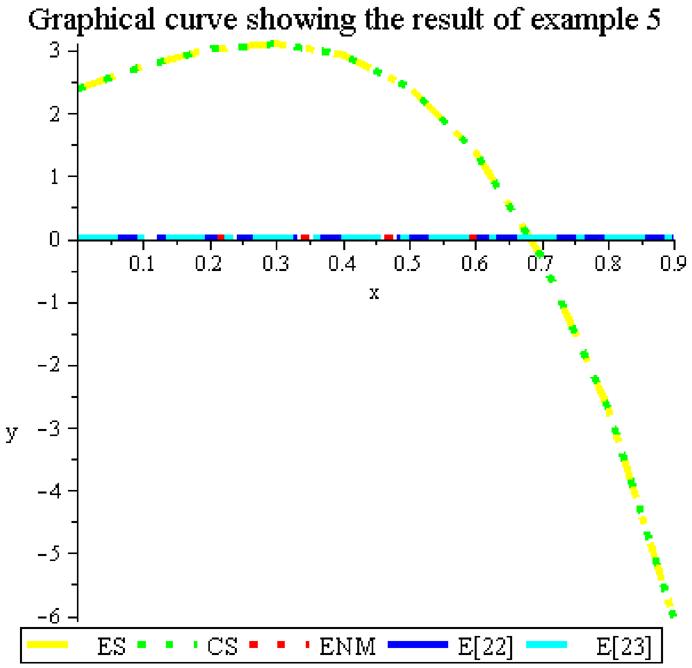

Figure 4: Textual graph of table 6

The new method was employed to simulate five different oscillatory differential equations. The results, as presented in Tables 1 to 6, clearly demonstrate that the new method outperforms the ones it was compared with.

Furthermore, the simulation results depicted in Figures 1 to 6 confirm the effectiveness of the new method in simulating oscillatory differential equations.

The application of the new method to simulate oscillatory differential equations in harmonic motion revealed superior convergence compared to the method described in reference [19], as evident in Table 1 and Figure 1.
Table 2 provides a comparison of the new method with references [12, 13] when solving oscillatory differential equations in spring motion. This comparison sheds light on the influence of external force "F" on the system's behavior.
Similarly, the new method was applied to simulate second-order oscillatory differential equations in mass dynamic motion, Betiss and Stiefel equations, and nonlinear oscillatory differential equations (examples 3 to 5). The results in Tables 3 to 6 and Figures 3 to 6 unequivocally demonstrate that the new method surpasses the methods described in references [12, 13, 20-23].
Conclusively, both the tabulated results in Tables 1 to 6 and the graphical representations in Figures 1 to 6 validate the efficiency and effectiveness of the new method in handling second-order oscillatory differential equations.

6 Conclusion

This research delved into the numerical approximation and practical application of oscillations in a moving mass. The new method was developed based on power series polynomials, and its properties were rigorously analyzed. It has been found that the new method exhibits computational reliability superior to the methods considered for solving similar oscillatory differential equations.

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