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Perturbed Collocation Methods for the Solution of Higher Order Fractional Integro-differential Equations Boundary Value Problems

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Abstract
This article explores the use of two orthogonal polynomial approximation methods to derive numerical solutions for boundary value problems involving higher-order fractional integro-differential equations. We introduce a perturbed collocation approach that transforms these perturbed equations into systems of algebraic equations by employing standard collocation points. The resulting algebraic systems are solved using Newton-Raphson's method, implemented through MAPLE 18 software. Several numerical examples are provided to demonstrate the accuracy and reliability of this method. The findings indicate that the proposed approach is both accurate and efficient. Additionally, the results show a favorable comparison with those obtained by Zhang et al. using the Homotopy Analysis Method.

Keywords: Boundary value problems, Chebyshev polynomials, Fractional derivatives, Perturbation term, Power series polynomials, Perturbed Collocation Method, Newton Raphson method

1 Introduction

Boundary value problems can be effectively approximated using simple and efficient numerical methods. Problems involving the wave equation, such as determining normal modes, are often formulated as boundary value problems. Sturm-Liouville problems represent an important class of boundary value problems, and their analysis involves the eigenfunctions of a differential operator, as discussed by Fu et al.[2].

Zhang et al.[1] employed the Homotopy Analysis Method (HAM) to obtain numerical solutions for higher-order fractional integro-differential equations with boundary value problems. They reported that the numerical results are in good agreement with the exact solution and converge at higher-order approximations. Fadugba [3] presented the Mellin transform approach for solving fractional order equations, which is widely used in applied mathematics and technology. The Mellin transform of various forms of fractional calculus, including the Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral, Caputo fractional derivative, and the Miller-Ross sequential fractional derivative, were obtained.

Methods for solving integro-differential equations typically combine techniques for solving both integral and differential equations. Since closed-form solutions may not be feasible for most applications, numerical methods are employed to obtain approximations to the exact solutions. Abubakar & Taiwo [4], in their thesis "Computational methods for solving system of linear Volterra integral and integro-differential equations," reported the use of successive approximation methods and standard integral collocation to obtain numerical approximations. Uwaheren et al.[5] developed a perturbed collocation method for solving singular multi-order fractional differential equations of Lane-Emden Type, and found the proposed method to be efficient and yielding good results.

Other methods, such as power series, Chebyshev, and Legendre's polynomials used as basis functions, have been applied to obtain solutions for some higher-order integro-differential equations of both linear and nonlinear types. Akyaz & Sezer [6] used truncated Chebyshev polynomials to obtain approximate solutions of linear integro-differential equations, and the results showed that the method is consistent. Recently, Gelele et al. [7] used power series and Chebychev series approximation methods for solving higher-order linear Fredholm integro-differential equations using collocation methods. The results presented indicated that the method provides accurate results when compared with the exact solution.

Many problems from various sciences and engineering applications can be modeled by fractional integro-differential equations. Furthermore, most of these problems cannot be solved analytically, and finding good approximate solutions using numerical methods will be very helpful, as pointed out by Yang et al.[8]

Several numerical methods have been recently developed to solve fractional differential equations (FDEs) and fractional integro-differential equations (FIDEs). Yang et al. and Mittal & Nigam [8-14] applied collocation methods for solving the following: nonlinear fractional Langevin equation involving two fractional orders in different intervals and fractional Fredholm integro-differential equations. Chebyshev polynomials method is introduced in the literature [10-12] for solving multi-term fractional order differential equations and nonlinear Volterra and Fredholm integro-differential equations of fractional order. Ray [13] applied the variational iteration method for solving fractional integro-differential equations with nonlocal boundary conditions. Adomian decomposition method is introduced in Mittal & Nigam [14] and Wazwaz [15] for solving fractional diffusion equation and fractional integro-differential equations.

In this paper, we present a numerical solution approach for boundary value problems involving higher-order fractional integro-differential equations. To minimize higher error terms, we introduce a perturbation term into the model equation. We then apply Power series and Chebyshev polynomial approximations to derive efficient numerical solutions for the relevant problems. The resulting equations are collocated to form a system of algebraic equations, which allows us to solve for the unknown coefficients.

The structure of the paper is as follows: In Section 2, we introduce preliminary definitions and key properties. Section 3 provides the fundamental definitions and characteristics of power series polynomials and Chebyshev polynomials. In Sections 4 and 5, we apply these polynomials to perturbed higher-order fractional integro-differential equations using the standard collocation method, and we utilize a matrix operation solver to address the resulting systems of equations. Section 6 presents several numerical examples to illustrate the efficiency and accuracy of the proposed algorithm. Finally, we conclude the paper with remarks in Section 7.

Consider the general form of fractional integro-differential equation boundary value problems with Caputo derivative defined in Zhang [1] as follows

$$ {}_{a}^{C}D_{t}^{\alpha}u(t) = f(t) + \gamma u(t) + \lambda \int_{a}^{b} k(t,s)u(s)ds, \tag{1} $$

subject initial-boundary conditions

$$ u^{(i)}(a) = \mu_i, u^{(i)}(b) = \eta_i, i = 0(1)n \tag{2} $$

Where is the Caputo fractional derivative of order are given continuous smooth functions, is the unknown function to be determined, and are real constants.where ${}_{a}^{C}D_{t}^{\alpha}$ is the Caputo fractional derivative of order α, $f(t), k(t,s)$ are given continuous smooth functions, $u(s)$ is the unknown function to be determined, and a, b, λ, γ are real constants.

2 Preliminaries

Here, we give some basic definitions, theorem and properties of fractional calculus theory that can be used in understanding this paper.

Definition 2.1: The most common fractional operators are the Riemann-Liouville Fractional Integral (RLFI), the Riemann-Liouville Fractional Derivative (RLFD) and the Caputo Fractional Derivative (CFD) which are defined as follows:
Let $x:[\mathrm{a},\mathrm{b}]\rightarrow R$ be a function, let $\alpha > 0$ be a real number, and let $n=\lceil\alpha\rceil$, where α denotes the smallest integer greater than or equal to α . For $t\in[a,b]$, defined

$$ {}_aI_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)d $$

$$ {}_tI_b^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}x(\tau)d $$

$$ {}_aD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}x(\tau)d\tau $$

$$ {}_tD_b^{\alpha}x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1}x(\tau)d\tau $$

$$ {}_a^CD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau $$

$$ {}_t^CD_b^{\alpha}x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}x^{(n)}(\tau)d\tau $$

where $n=\lceil\alpha\rceil+1$ for the definitions of Riemann-Liouville fractional derivatives, and for the Caputo fractional derivatives.
Definition 2.2: Let the function $\Gamma:(0,\infty)\rightarrow R$, defined by

$$ \Gamma(n):=\int_0^{\infty} e^{-t}t^{n-1}dt, n>0. \tag{3} $$

Definition 2.3: Beta function is defined in terms of gamma function as

$$ B(n,m)=\int_0^1 t^{n-1}(1-t)^{m-1}dt=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}, n,m\in R^{+}. \tag{4} $$

Theorem 2.1: Let $\Phi\in H_{\mu}[a,b]$ for some $\mu\in[0,1]$ and let $0<n<1.$ Then

$$ J_a^{\alpha}\Phi(t)=\frac{\Phi(a)}{\Gamma(\alpha+1)}(x-a)^{\alpha}+\Psi(t), \tag{5} $$

with some function $\Psi(t)=O\left((x-a)^{\mu+\alpha}\right).$

3 Polynomial Approximate solutions

An approximate solution of the truncated Chebyshev and Power series orthogonal polynomials can be used to obtain the approximations of (1) and (2) as defined in the form

$$u_n(t) = \sum_{i=0}^{n} a_i \Phi_i(t), 0 < n < \infty. \tag{6}$$

and

$$\Phi_i(t) = \begin{cases} P_i(t) \\ T_i(t) \end{cases}, \tag{7}$$

where τ is the unknown to be determined and $P(t)$ and $T(t)$ are power series and Chebyshev orthogonal polynomials respectively.

The $r-th$ degree chebyshev polynomial of the first kind valid in [a,b] is defined as

$$T_r(x) = \cos\left\{ \mathrm{r}\cos^{-1}\left(\frac{2x-2a}{b-a} - 1 \right) \right\} \equiv \sum_{k=0}^{r} C_k^{(r)} x^k, \tag{8}$$

with

$$C_r^{(r)} = 2^{2r-1}(b-a)^{r}. \tag{9}$$

For this purpose, we consider $a = 0$ and $b = 1$, so that (8) becomes

$$T_r(x) = \cos\{\mathrm{r}\cos(2\mathrm{x}-1)\} \equiv \sum_{k=0}^{r} C_k^{(r)} x^k, \tag{10}$$

$$T_0(x) = 1, T_1(x) = (2x-1) \tag{11}$$

This satisfies the recurrence relation

$$T_{r+1} = 2(2x-1)T_r(x) \quad T_{r \quad 1}(x), r = 1,2,3,\ldots \tag{12}$$

4 Perturbed Collocation Method by Power Series

Sequel to the theorem 2.1, we established the following corollary

Corollary 3.1 Let $Q([t_0, t_f], R)$ be defined by $Q(s) = (s - t_0)^{\beta}$ for some $\beta > -1$ and $\alpha > 0$, and let $\alpha, \beta \in R_+$ and $s \in [t_0, t_f]$. Then,

$$\frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} Q'(s) ds = \frac{\Gamma(\beta+1)}{\Gamma(1-\alpha+\beta)} (t-t_0)^{\alpha+\beta}. \tag{13}$$

Proof: Using left Caputo fractional derivative for $n = 1$, we have

$$_{t_0}^{C} D_t^{\alpha} Q(s) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{\alpha} Q'(s) ds \tag{14}$$

Substituting $Q'(s) = \beta(s-t_0)^{\beta-1}$ into (14), to obtain

$$\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}\beta(s-t_0)^{\beta-1}ds. \tag{15}$$

Let $x=\frac{(s-t_0)}{(t-t_0)}$, so that $s=x(s-t_0)+t_0$ and $(t-t_0)dx=ds$, then

$$\frac{1}{\Gamma(1-\alpha)} \quad =\int_{t_0}^{t}(t-s)^{\ \alpha}\beta(s-t_0)^{\beta-1}ds \tag{16}$$

$$\frac{\beta}{\Gamma(1-\alpha)} \quad =\int_{0}^{1}[(t-t_0)-x(t-t_0)]^{-\alpha}[x(t-t_0)]^{\beta-1}(t-t_0)dx$$

$$\frac{\beta(t-t_0)^{-\alpha+\beta}}{\Gamma(1-\alpha)} \quad =\int_{0}^{1}(1-x)^{-\alpha}x^{\beta-1}dx.$$

Using Beta function $\int_{0}^{1}(1-t)^{\alpha-1}t^{\beta-1}dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, we have

$$\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-s)^{-\alpha}(s-t_0)^{\beta-1}ds=\frac{\Gamma(\beta+1)}{\Gamma(1-\alpha+\beta)}(t-t_0)^{-\alpha+\beta}. \tag{17}$$

■

Now considering equation (1) together with the perturbation term of the form

$${}_{a}^{C}D_{t}^{\alpha}u_n(t)=f(t)+\gamma u_n(t)+\lambda\int_{a}^{b}k(t,s)u_n(s)ds+H_n(t), \tag{18}$$

where $a<t<b$,

$$H_n(t)=\sum_{r=0}^{n-1}\tau_{m-r}T_{n\ \ m+r+1}(t) \tag{19}$$

Now, substituting (6) into (18) and simplify to obtain

$$D_{t}^{\alpha}\left(a_0+a_1t+a_2t^2+a_3t^3+\ldots+a_nt^n\right)=f(t) \tag{20}$$
$$+\gamma\left(a_0+a_1t+a_2t^2+a_3t^3+\ldots+a_nt^n\right)$$
$$+\lambda\int_{a}^{b}k(t,s)\left(a_0+a_1s+a_2s^2+a_3s^3+\ldots+a_ns^n\right)ds$$
$$+\tau_1T_n(t)+\tau_2T_{n\ \ 1}(t)+\tau_3T_{n\ \ 2}(t)+\ldots$$

simplifying (20) further to obtain

$$\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t^{n-\alpha}-\gamma-\lambda\int_a^b k(t,s)ds\right)a_0 \tag{21}$$

$$+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t^{n-\alpha}-\gamma t-\lambda\int_a^b k(t,s)sds\right)a_1$$

$$+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t^{n-\alpha}-\gamma t^2-\lambda\int_a^b k(t,s)s^2ds\right)a_2$$

$$+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t^{n-\alpha}-\gamma t^3-\lambda\int_a^b k(t,s)s^3ds\right)a_3$$

$$+\ldots+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t^{n-\alpha}-\gamma t^n-\lambda\int_a^b k(t,s)s^nds\right)a_n$$

$$\tau_1T_n(t)-\tau_2T_{n-1}(t)-\tau_3T_{n-2}(t)-\ldots=f(t)$$

Therefore, the time interval $[a,b]$ is divided into N equally spaced (sub-intervals). Let $t_j = a + hj$, where $h=\frac{b-a}{N}$ and $j = 0(1)N$, then, there is need to construct the associated system of algebraic equations in a manner that require less computational time and give efficient results.

Now, collocating (21) at the node of t_j, we have

$$\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t_j^{n-\alpha}-\gamma-\lambda\int_a^b k(t,s_j)s\,ds\right)a_0 \tag{22}$$

$$+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t_j^{n-\alpha}-\gamma t_j-\lambda\int_a^b k(t,s_j)s_j\,ds\right)a_1$$

$$+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t_j^{n-\alpha}-\gamma t_j^2-\lambda\int_a^b k(t,s_j)s_j^2\,ds\right)a_2$$

$$+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t_j^{n-\alpha}-\gamma t_j^3-\lambda\int_a^b k(t,s_j)s_j^3\,ds\right)a_3$$

$$+\ldots+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}t_j^{n-\alpha}-\gamma t_j^n-\lambda\int_a^b k(t,s_j)s_j^n\,ds\right)a_n$$

$$\tau_1T_n(t_j)-\tau_2T_{n-1}(t_j)-\tau_3T_{n-2}(t_j)-\ldots=f(t_j).$$

Thus (21) gives rise to (N+3) algebraic linear equations in (N+3) unknown constants $(a_0,a_1,\ldots,a_N,\tau_1,\tau_2,\tau_3,\ldots)$ together with the extra equations obtained from the boundary conditions. Altogether, we have (N+3) algebraic linear equations in (N+3) unknown constants which are then solved by Maple 18 software to obtain the values of the unknown constants.

5 Perturbed Collocation Method by Chebyshev Polynomials

In order to apply this method, we substitute an approximate solution (6) into a slightly perturbed equation (18) to gives

$$D_t^\alpha\left(a_0T_0(t)+a_1T_1(t)+a_2T_2(t)+a_3T_3(t)+\ldots+a_nT_n(t)\right)=f(t) \tag{23}$$
$$+\gamma\left(a_0T_0(t)+a_1T_1(t)+a_2T_2(t)+a_3T_3(t)+\ldots+a_nT_n(t)\right)$$

$$+\lambda\int_a^b k(t,s)\left(a_0T_0(s)+a_1T_1(s)+a_2T_2(s)+\ldots+a_nT_n(s)\right)ds$$
$$+\tau_1T_n(t)+\tau_2T_{n-1}(t)+\tau_3T_{n-2}(t)+\ldots$$

simplifying (23) further to obtain

$$
\begin{aligned}
&\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_0^{n-\alpha}(t)-\gamma T_0(t)-\lambda\int_a^b k(t,s)T_0(s)ds\right)a_0 \\
&+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_1^{n-\alpha}(t)-\gamma T_1(t)-\lambda\int_a^b k(t,s)T_1(s)ds\right)a_1 \\
&+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_2^{n-\alpha}(t)-\gamma T_2(t)-\lambda\int_a^b k(t,s)T_2(s)ds\right)a_2 \\
&+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_3^{n-\alpha}(t)-\gamma T_3(t)-\lambda\int_a^b k(t,s)T_3(s)ds\right)a_3 \\
&+\ldots+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_0^{n-\alpha}(t)-\gamma T_n(t)-\lambda\int_a^b k(t,s)T_n(s)\right)a_n \\
&\tau_1 T_n(t)-\tau_2 T_{n-1}(t)-\tau_3 T_{n-2}(t)-\ldots= f(t)
\end{aligned} \tag{24}
$$

Now, collocating (24) at the node of t_j , we have

$$
\begin{aligned}
&\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_0^{n-\alpha}(t_j)-\gamma T_0(t_j)-\lambda\int_a^b k(t,s_j)T_0(s_j)ds\right)a_0 \\
&+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_1^{n-\alpha}(t_j)-\gamma T_1(t_j)-\lambda\int_a^b k(t,s_j)T_1(s_j)ds\right)a_1 \\
&+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_2^{n-\alpha}(t_j)-\gamma T_2(t_j)-\lambda\int_a^b k(t,s_j)T_2(s_j)ds\right)a_2 \\
&+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_3^{n-\alpha}(t_j)-\gamma T_3(t_j)-\lambda\int_a^b k(t,s_j)T_3(s_j)ds\right)a_3 \\
&+\ldots+\left(\frac{\Gamma(n+1)}{\Gamma(1-\alpha+n)}T_n^{n-\alpha}(t_j)-\gamma T_n(t_j)-\lambda\int_a^b k(t,s_j)T_n(s_j)ds\right)a_n \\
&-\tau_1 T_n(t_j)-\tau_2 T_{n-1}(t_j)-\tau_3 T_{n-2}(t_j)-\ldots= f(t_j).
\end{aligned} \tag{25}
$$

6 Numerical Examples

Problem 1: Consider the Fractional order integro-differential equation defined by Zhang et al [1]. as

$$D_t^{\alpha}u(t)=t\left(1+e^t\right)+3e^t+u(t)\int_0^t u(s)ds, 0<t<1, 3<\alpha\leq 4, \tag{26}$$

subject to boundary conditions

$$u(0)=1, u''(0)=2, u(1)=1+e, u''(1)=3e. \tag{27}$$

Comparing with equations (1) and (2), we have

$f(t)=t\left(1+e^t\right)+3e^t, \gamma=1, \lambda=\ \ 1, k(t,s)=1, \mu_0=1, \mu_2=2, \eta_0=1+e$ and $\eta_2=3e.$ (28)

Therefore, consider forth degree $(n=4)$ approximation, we have from approximate solution

$$u_4=\sum_{i=0}^{4}a_it^i=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4. \tag{29}$$

Using power series polynomial approximate solution, we substitute equation (26) into collocation equation (25), and together with the extra equations obtained from boundary conditions, gives the following matrix equation of the form $Ax=b$ as

$$\begin{pmatrix} 0.50000 & 0.2500 & -0.83333 & -0.25000 & -298645073 & 0.33333 & -0.43321 & 0.55321 & 0.75321 \\ 0.87500 & 0.12500 & -0.73333 & -0.25000 & -398645073 & 0.33300 & -0.43320 & 0.55320 & 0.75320 \\ 0.95000 & 0.56500 & -0.53333 & -0.85000 & -498645073 & 0.63300 & -0.63300 & 0.65320 & 1.00000 \\ 0.98500 & 0.25000 & -0.73333 & -0.72545 & -598645073 & 0.67300 & -0.43320 & 0.89532 & 0.65532 \\ 0.54500 & 0.67250 & -0.33300 & -0.725070 & -698645070 & 0.73300 & -0.87300 & 0.45500 & 0.24500 \\ 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & -96 & 320 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 96 & 320 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} -6.70524 \\ -8.07055 \\ -9.17052 \\ -13.25245 \\ -17.37052 \\ 1 \\ 1+e \\ 2 \\ 3e \end{pmatrix}$$

Again, consider sixth degree $(n=6)$ approximation, we have from approximate solution

$$u_6=\sum_{i=0}^{6}a_it^i=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5+a_6t^6. \tag{30}$$

Using Chebyshev polynomial approximate solution, we substitute equation (28) into collocation equation (25), and together with the extra equations obtained from boundary conditions, gives also the system of equation of the form $Ax=b$.

Problem 2: Consider the Fractional order integro-differential equation defined by Zhang et al [1]. as

$$D_t^{\alpha}u(t)=1+(5+t)e^t+u(t)\int_0^t u(s)ds, 0<t<1, 5<\alpha\leq 6, \tag{31}$$

subject to boundary conditions

$$u(0)=0, u''(0)=2, u^{(iv)}(0)=4, u(1)=e, u''(1)=3e u^{(iv)}=5e. \tag{32}$$

Comparing with equations (1) and (2), we have

$f(t)=1+(5+t)e^t, \gamma=1, \lambda= \ 1, k(t,s)=1, \mu_0=0, \mu_2=2, \mu_4=4, \eta_0=1+e, \eta_2=3e$ and $\eta_4=5e.$

6 Conclusion

This article presents a numerical solution approach for higher-order fractional integro-differential equations with boundary value problems. Our methods are based on Power series polynomial and Chebyshev polynomial approximations, which reduce the perturbed higher-order fractional integro-differential equation to a set of linear algebraic equations. These equations can be easily solved using the standard collocation method and computer implementation. The results obtained demonstrate that both the Power series polynomial and Chebyshev polynomial methods can effectively handle these types of problems, as evident from the tables of results. The findings compare favorably with the work of Zhang et al. [1], who used the Homotopy Analysis Method. In the first problem, varying the order of α and β and the degree of approximations from 4 up to 6 shows that as the degree of approximation increases, the accuracy of the results improves. Similarly, in the second problem, the results obtained at higher degrees of approximation yield better outcomes. For future studies, we plan to investigate further the existence and uniqueness of perturbed fractional integro-differential equations with boundary value problems. Additionally, we aim to consider the numerical solution of systems of these problems. We welcome comments and feedback from fellow readers to enhance our research.

Conflict of Interest

The authors declared that there is no conflict of interest.

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References

[1] X. Zhang, B. Tang & Y. He (2011). Homotopy analysis method for higher order fractional integrodifferential equations. *Journal of Computer and Mathematics with Applications*, **16**, 3194. https://doi.org/10.1016/j.camwa.2011.08.032.

[2] Z. J. Fu, W. Chen, & H. T. Yang (2013). Boundary particle method for Laplace transformed time fractional diffusion equations. *Journal of Computational Physics*, **235,** 52.

[3] S. E. Fadugba (2019). Solution of Fractional Order Equations in the Domain of the Mellin Transform, *Journal of the Nigerian Society of Physical Sciences* **4,** 138. https://doi.org/10.46481/jnsps.2019.31

[4] A. Abubakar & O. A Taiwo (2014). Integral collocation approximation methods for the numerical solution of high-orders linear Fredholm-Volterra integro-differential equations. *American Journal of Computational and Applied Mathematics*, **4**, 111.

[5] O. A. Uwaherena, A. F. Adebisi, & O. A. Taiwo (2020). Perturbed Collocation Method For Solving Singular Multi-order Fractional Differential Equations of Lane-Emden Type *Journal of the Nigerian Society of Physical Sciences*, **2,** 148.

[6] A Akyaz & M. Sezer (1999). A Chebyshev collocation method for the solution of linear integro-differential equations. *International Journal of Computer Mathematics,* **72**, 491.

[7] O. A Gegele, O. P Evans & D. Akoh (2014). Numerical solutions of higher order linear Fredholm integro-differential equations. *American Journal of Engineering Research*, **8,** 243.

[8] Y. Yang, Y. Chen, & Y. Huang. (2014).“Spectral-collocation method for fractional Fredholm integro-differential equations". *Journal of the Korean Mathematical Society*, **51,** 203.

[9] G. Ajileye, A. A. James, A. M. Ayinde & T. Oyedepo (2022). Collocation approach for the computational solution of Fredholm-Volterra Freactional order of integro-differential equations. *Journal of the Nigerian Society of Physical Sciences*, **4,** 834. https://doi.org/10.46481/jnsps.2022.834.

[10] E. H. Doha, A. H. Bhrawy, & S. S. Ezz-Eldien (2011). Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. *Applied Mathematical Modeling,* **35,** 5662.

[11] S. Irandoust-pakchin, H. Kheiri, & S. Abdi-mazraeh. (2013). Chebyshev cardinal functions: an effective tool for solving nonlinear Volterra and Fredholm integro-differential equations of fractional order. *Iranian Journal of Science and Technology Transaction A: Science,* **37,** 53.

[12] S. Irandoust-pakchin & S. Abdi-Mazraeh (2013). Exact solutions for some of the fractional integro-differential equations with the nonlocal boundary conditions by using the modification of He's variational iteration method. *International Journal of Advanced Mathematical Sciences,* **1,** 139.

[13] S. Saha Ray (2009). Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method. *Communications in Nonlinear Science and Numerical Simulation*, **14,** 1295.

[14] R. C. Mittal & R. Nigam (2008). Solution of fractional integro-differential equations by Adomain decomposition method. *International Journal of Applied Mathematics and Mechanics,* **4,** 87.

[15] A. Wazwaz (2011). *Linear and Nonlinear Integral Equations, Methods and Applications*, Higher Educational Press, Beijing and Springer-Verlag Berlin Heidlberg.

[16] R. Almedia, & D. F. M. Torres (2011). Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives". *Commun Nonlinear Sci Numer Simul,* **16** (2011) 1490.

[17] R. Almeida, & D. F. M. Torres (2014). A discrete method to solve fractional optimal control problems". *Nonlinear Dyn*, **80,** 1811.

[18] N. Kurt, & S. Mehmet (2008). Polynomial solution of high-order linear fredholm integro-differential equations with constant coefficients. *Journal of the Franklin Institute, Elsevier*. **2,** 839. doi:10.1016/j.jfranklin. 345.