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Derivation and Analysis of Block Hybrid Method for Solving Initial Value Problems in Oscillatory Differential Equations

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Abstract

The Block Hybrid Method is a numerical technique for solving ordinary differential equations (ODEs), particularly effective for stiff and oscillatory systems. This paper introduces a new method designed to handle challenges posed by equations like the Malthusian Growth Model and Prothero-Robinson equation, which are difficult to solve using conventional methods due to stiffness and rapid oscillations. Derived using power series approximation, the method is analyzed for order, error constant, consistency, and zero stability, proving to be convergent, consistent, and zero-stable. Numerical examples demonstrate its superior accuracy and stability compared to existing methods, making it a valuable tool for solving complex initial value problems in real-world applications.

Keywords: Numerical Methods, Oscillatory Differential Equations, Computational Efficiency, Stability Analysis, Block Hybrid Method

1 Introduction

To tackle real-world challenges across engineering, biological sciences, physical sciences, electronics, and other disciplines, researchers frequently encounter initial value problems, as noted by [1]. Many practical problems in engineering and science are initially formulated as differential equations before resolution. These equations typically involve derivatives, establishing a connection between an independent variable, a dependent variable and one or more differential coefficients concerning x [2, 3]. Although discrete hybrid methods have been devised, their superior accuracy compared to conventional linear multi-step methods of identical step-size, as highlighted by Lambert in [4], has not yet garnered the anticipated level of attention. This study aims to address initial value problems (IVPs) structured as:

$$y'(x) = f(x, y), \qquad a \le x \le b, \qquad y(a) = y_0 \tag{1.1}$$

The Block Hybrid Method is a numerical technique used to solve ordinary differential equations (ODEs), particularly stiff systems, with higher accuracy and efficiency. It combines the advantages of both multistep methods and Runge-Kutta methods, providing a robust solution for a wide range of differential equation problems [4]. The Block Hybrid Method offers a versatile and efficient approach to solving ordinary differential equations, especially in scenarios involving stiff systems, making it a valuable tool in the numerical analysis toolkit [3, 5].

The Malthusian Growth Model, Prothero-Robinson equation and highly stiff oscillatory differential equations are important examples of initial value problems (IVPs) in numerical analysis. The Malthusian Growth Model, introduced by Thomas Malthus, describes exponential population growth under the assumption of unlimited resources, where the rate of change in population is proportional to the current population size [6, 7]. This model is fundamental in understanding population dynamics and is often used in ecology and economics. Its simplicity, represented by the first-order differential equation $\frac{d\rho}{dt}=\kappa\rho$, where ρ is the population size and κ is the growth rate, allows for easy analytical solutions. However, more complex biological systems require extensions of this model to capture factors like resource limitations or carrying capacities, introducing nonlinearities and complexity into IVP formulations [8, 9].

The Prothero-Robinson equation and stiff oscillatory differential equations present significantly more challenging problems. The Prothero-Robinson equation highlights the difficulties posed by stiffness, a property of differential equations where certain components evolve much faster than others, leading to numerical instabilities [10, 11]. This equation is used to test the robustness of numerical methods, especially in systems involving multi-scale phenomena. Highly stiff oscillatory differential equations, often encountered in systems involving mechanics, physics and engineering, exhibit rapid oscillations that make their numerical solution particularly difficult [12, 13]. Traditional numerical methods struggle with accuracy and stability in such cases, necessitating advanced techniques like implicit methods or specialized solvers. Together, these models underscore the importance of selecting appropriate methods for solving IVPs, particularly when dealing with systems with diverse behaviors such as exponential growth, oscillations, and stiffness [14, 15].

2 Derivation of the Block Hybrid Method

In this section, we will utilize the concepts introduced in preceding sections to construct block hybrid method aimed at solving first-order initial value problems in oscillatory differential equation expressed in the form (1.1). The power series as an approximate solution of the form;

$$y(x)=h\sum_{i=0}^{m+n-1}\alpha_j\chi^i \tag{2.1}$$

is consider deriving the method, where m and n are distinct point of interpolation and collocation [9].

2.2 Formulation of the Block Hybrid Method

The power series polynomial (2.1) is consider as an approximate solution of (1.1).
Differentiate (2.1) once to yield,

$$\frac{dy}{dx}=h\sum_{i=0}^{m+n-1}i\alpha_j\chi^{i-1} \tag{2.2}$$

Where $\alpha\in\Re$ for $i=0\left(\frac{1}{3}\right)2$ and $y(x)$ is continuously differential. Let the solution of (1.1) be sought on the integration interval $[a,b]$ with a constant step-size h defined by $h=\chi_{n+1}-\chi_n$, $n=0,1,\cdots,N$.
Substituting equation (2.2) into (1.1) gives,

$$f(x,y)=h\sum_{i=0}^{m+n-1}i\alpha_j\chi^{i-1} \tag{2.3}$$

We interpolate equation (2.1) at point, x_{n+m}, $m=\frac{1}{3}$ and collocate equation (2.3) at points

x_{n+n}, $n=0,\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\frac{5}{3},2$ to give,

$$A\chi=U \tag{2.4}$$

Where

$$A=[a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7]^T,\ U=\left[y_{n+\frac{1}{3}},\ f_n,\ f_{n+\frac{1}{3}},\ f_{n+\frac{2}{3}},\ f_{n+1},\ f_{n+\frac{4}{3}},\ f_{n+\frac{5}{3}},\ f_{n+2}\right]^T$$

$$X=\begin{bmatrix} \chi^0_{n+\frac{1}{3}} & \chi^1_{n+\frac{1}{3}} & \chi^2_{n+\frac{1}{3}} & \chi^3_{n+\frac{1}{3}} & \chi^4_{n+\frac{1}{3}} & \chi^5_{n+\frac{1}{3}} & \chi^6_{n+\frac{1}{3}} & \chi^7_{n+\frac{1}{3}} \\ 0 & 1\chi^0_n & 2\chi^1_n & 3\chi^2_n & 4\chi^3_n & 5\chi^4_n & 6\chi^5_n & 7\chi^6_n \\ 0 & 1\chi^0_{n+\frac{1}{3}} & 2\chi^1_{n+\frac{1}{3}} & 3\chi^2_{n+\frac{1}{3}} & 4\chi^3_{n+\frac{1}{3}} & 5\chi^4_{n+\frac{1}{3}} & 6\chi^5_{n+\frac{1}{3}} & 7\chi^6_{n+\frac{1}{3}} \\ 0 & 1\chi^0_{n+\frac{2}{3}} & 2\chi^1_{n+\frac{2}{3}} & 3\chi^2_{n+\frac{2}{3}} & 4\chi^3_{n+\frac{2}{3}} & 5\chi^4_{n+\frac{2}{3}} & 6\chi^5_{n+\frac{2}{3}} & 7\chi^6_{n+\frac{2}{3}} \\ 0 & 1\chi^0_{n+1} & 2\chi^1_{n+1} & 3\chi^2_{n+1} & 4\chi^3_{n+1} & 5\chi^4_{n+1} & 6\chi^5_{n+1} & 7\chi^6_{n+1} \\ 0 & 1\chi^0_{n+\frac{4}{3}} & 2\chi^1_{n+\frac{4}{3}} & 3\chi^2_{n+\frac{4}{3}} & 4\chi^3_{n+\frac{4}{3}} & 5\chi^4_{n+\frac{4}{3}} & 6\chi^5_{n+\frac{4}{3}} & 7\chi^6_{n+\frac{4}{3}} \\ 0 & 1\chi^0_{n+\frac{5}{3}} & 2\chi^1_{n+\frac{5}{3}} & 3\chi^2_{n+\frac{5}{3}} & 4\chi^3_{n+\frac{5}{3}} & 5\chi^4_{n+\frac{5}{3}} & 6\chi^5_{n+\frac{5}{3}} & 7\chi^6_{n+\frac{5}{3}} \\ 0 & 1\chi^0_{n+1} & 2\chi^1_{n+1} & 3\chi^2_{n+1} & 4\chi^3_{n+1} & 5\chi^4_{n+1} & 6\chi^5_{n+1} & 7\chi^6_{n+1} \end{bmatrix}$$

Solving (2.4), for $\alpha_i,\, i=0\left(\frac{1}{3}\right)2$ and replacing back into (2.1) gives a linear block scheme as

$$y(t)=\alpha_{\frac{1}{3}}(t)y_{n+\frac{1}{3}}+h\begin{bmatrix}\beta_0(t)f_n+\beta_{\frac{1}{3}}(t)f_{n+\frac{1}{3}}+\beta_{\frac{2}{3}}(t)f_{n+\frac{2}{3}} \\ +\beta_1(t)f_{n+1}+\beta_{\frac{4}{3}}(t)f_{n+\frac{4}{3}}+\beta_{\frac{5}{3}}(t)f_{n+\frac{5}{3}}+\beta_2(t)f_{n+2}\end{bmatrix} \tag{2.5}$$

Where

$\alpha_{\frac{1}{3}}=1$

$\beta_0=\frac{1}{181440}\left(-19087+66918t-93501t^2+64476t^3-21870t^4+2916t^5\right)(3t-1)^2$

$\beta_{\frac{1}{3}}=-\frac{1}{7560}\left(-2713-8139t+43623t^2-66447t^3+4704t^4-16281t^5+2187t^6\right)(3t-1)$

$\beta_{\frac{2}{3}}=\frac{1}{60480}\left(15487+92922t-262251t^2+243756t^3-98010t^4+14580t^5\right)(3t-1)^2$

$\beta_1=-\frac{1}{11340}\left(+2344+14064t-50112t^2+52812t^3-23085t^4+3645t^5\right)(3t-1)^2$

$\beta_{\frac{4}{3}}=\frac{1}{60480}\left(6737+40422t-15830t^2+183276t^3-86670t^4+14580t^5\right)(3t-1)^2$

$\beta_{\frac{5}{3}}=-\frac{1}{7560}\left(263+1578t-6507t^2+7992t^3-4050t^4+729t^5\right)(3t-1)^2$

$\beta_1=\frac{1}{181440}\left(863+5178t-22059t^2+28188t^3-15066t^4+2916t^5\right)(3t-1)^2$

for $t=\frac{(\chi-\chi_n)}{h}$.

Evaluating (2.5) at non-interpolating points to gives

$$
\left.\begin{aligned}
y_n &= y_{n+\frac{1}{3}} + \psi_{01} f_n + \psi_{02} f_{n+\frac{1}{3}} + \psi_{03} f_{n+\frac{2}{3}} + \psi_{04} f_{n+1} + \psi_{05} f_{n+\frac{4}{3}} + \psi_{06} f_{n+\frac{5}{3}} + \psi_{07} f_{n+2} \\
y_{n+\frac{2}{3}} &= y_{n+\frac{1}{3}} + \psi_{11} f_n + \psi_{12} f_{n+\frac{1}{3}} + \psi_{13} f_{n+\frac{2}{3}} + \psi_{14} f_{n+1} + \psi_{15} f_{n+\frac{4}{3}} + \psi_{16} f_{n+\frac{5}{3}} + \psi_{17} f_{n+2} \\
y_{n+1} &= y_{n+\frac{1}{3}} + \psi_{21} f_n + \psi_{22} f_{n+\frac{1}{3}} + \psi_{23} f_{n+\frac{2}{3}} + \psi_{24} f_{n+1} + \psi_{25} f_{n+\frac{4}{3}} + \psi_{26} f_{n+\frac{5}{3}} + \psi_{27} f_{n+2} \\
y_{n+\frac{4}{3}} &= y_{n+\frac{1}{3}} + \psi_{31} f_n + \psi_{32} f_{n+\frac{1}{3}} + \psi_{33} f_{n+\frac{2}{3}} + \psi_{34} f_{n+1} + \psi_{35} f_{n+\frac{4}{3}} + \psi_{36} f_{n+\frac{5}{3}} + \psi_{37} f_{n+2} \\
y_{n+\frac{5}{3}} &= y_{n+\frac{1}{3}} + \psi_{41} f_n + \psi_{42} f_{n+\frac{1}{3}} + \psi_{43} f_{n+\frac{2}{3}} + \psi_{44} f_{n+1} + \psi_{45} f_{n+\frac{4}{3}} + \psi_{46} f_{n+\frac{5}{3}} + \psi_{47} f_{n+2} \\
y_{n+1} &= y_{n+\frac{1}{3}} + \psi_{51} f_n + \psi_{52} f_{n+\frac{1}{3}} + \psi_{53} f_{n+\frac{2}{3}} + \psi_{54} f_{n+1} + \psi_{55} f_{n+\frac{4}{3}} + \psi_{56} f_{n+\frac{5}{3}} + \psi_{57} f_{n+2}
\end{aligned}\right\} \qquad (2.6)
$$

Evaluating (2.6) at $t = \left[\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right]$ points to gives discrete block scheme of the form:

$$
\left.\begin{aligned}
y_{n+\frac{1}{3}} &= y_n + \Omega_{11} f_n + \Omega_{12} f_{n+\frac{1}{3}} + \Omega_{13} f_{n+\frac{2}{3}} + \Omega_{14} f_{n+1} + \Omega_{15} f_{n+\frac{4}{3}} + \Omega_{16} f_{n+\frac{5}{3}} + \Omega_{17} f_{n+2} \\
y_{n+\frac{2}{3}} &= y_n + \Omega_{21} f_n + \Omega_{22} f_{n+\frac{1}{3}} + \Omega_{23} f_{n+\frac{2}{3}} + \Omega_{24} f_{n+1} + \Omega_{25} f_{n+\frac{4}{3}} + \Omega_{26} f_{n+\frac{5}{3}} + \Omega_{27} f_{n+2} \\
y_{n+1} &= y_n + \Omega_{31} f_n + \Omega_{32} f_{n+\frac{1}{3}} + \Omega_{33} f_{n+\frac{2}{3}} + \Omega_{34} f_{n+1} + \Omega_{35} f_{n+\frac{4}{3}} + \Omega_{36} f_{n+\frac{5}{3}} + \Omega_{37} f_{n+2} \\
y_{n+\frac{4}{3}} &= y_n + \Omega_{41} f_n + \Omega_{42} f_{n+\frac{1}{3}} + \Omega_{43} f_{n+\frac{2}{3}} + \Omega_{44} f_{n+1} + \Omega_{45} f_{n+\frac{4}{3}} + \Omega_{46} f_{n+\frac{5}{3}} + \Omega_{47} f_{n+2} \\
y_{n+\frac{5}{3}} &= y_n + \Omega_{51} f_n + \Omega_{52} f_{n+\frac{1}{3}} + \Omega_{53} f_{n+\frac{2}{3}} + \Omega_{54} f_{n+1} + \Omega_{55} f_{n+\frac{4}{3}} + \Omega_{56} f_{n+\frac{5}{3}} + \Omega_{57} f_{n+2} \\
y_{n+1} &= y_n + \Omega_{61} f_n + \Omega_{62} f_{n+\frac{1}{3}} + \Omega_{63} f_{n+\frac{2}{3}} + \Omega_{64} f_{n+1} + \Omega_{65} f_{n+\frac{4}{3}} + \Omega_{66} f_{n+\frac{5}{3}} + \Omega_{67} f_{n+2}
\end{aligned}\right\} \qquad (2.7)
$$

3 Analysis of Basic Properties of the Block Hybrid Method

The necessary and sufficient conditions for new method and their associated block method are analyzed to establish their validity. These properties include; order and error constant, consistency, zero-stability and region of absolute stability.

3.1 Order and Error Constant

This subsection establishes the linear operator $\ell[y(x_i);h]$ associated with the newly derived method.

Definition 3.1

A linear multistep method is of order p if it satisfies the condition

$$c_0 = c_1 = c_2 = c_3 = \cdots = c_p = c_{p+1} = 0,\ c_{p+2} \neq 0,$$

Where

$$
\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k} \alpha_j \\
c_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j\right) \\
&. \\
&. \\
&. \\
c_p &= \sum_{j=0}^{k} \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!}\left(j^{p-1}\beta_j\right)\right],\ p = 2, 3, \cdots, q+1
\end{aligned}\right\} \qquad (3.1)
$$

The parameter $c_{p+2} \neq 0$ is referred to as the error constant with the local truncation error defined as $x_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) + c_{p+3} h^{p+3} y^{(p+3)}(x_n) + c_{p+4} h^{p+4} y^{(p+4)}(x_n) + 0(h^{p+5})$

$$\left[\begin{array}{l}
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{3}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\frac{2713}{7560}\left(\frac{1}{3}\right) - \frac{15487}{60480}\left(\frac{2}{3}\right) - \frac{586}{2835}(1) - \frac{6737}{60480}\left(\frac{4}{3}\right) + \frac{263}{7560}\left(\frac{5}{3}\right) - \frac{863}{181440}(2)\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{2}{3}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\frac{94}{189}\left(\frac{1}{3}\right) - \frac{11}{3780}\left(\frac{2}{3}\right) - \frac{332}{2835}(1) - \frac{269}{3780}\left(\frac{4}{3}\right) + \frac{22}{945}\left(\frac{5}{3}\right) - \frac{37}{11340}(2)\right] \\
\sum_{j=0}^{\infty} \frac{(1)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\frac{27}{56}\left(\frac{1}{3}\right) + \frac{387}{2240}\left(\frac{2}{3}\right) + \frac{34}{105}(1) - \frac{243}{2240}\left(\frac{4}{3}\right) + \frac{9}{280}\left(\frac{5}{3}\right) - \frac{29}{6720}(2)\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{4}{3}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\frac{464}{945}\left(\frac{1}{3}\right) + \frac{128}{945}\left(\frac{2}{3}\right) + \frac{1504}{2835}(1) - \frac{58}{945}\left(\frac{4}{3}\right) + \frac{16}{945}\left(\frac{5}{3}\right) - \frac{8}{2835}(2)\right] \\
\sum_{j=0}^{\infty} \frac{\left(\frac{5}{3}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\frac{275}{1512}\left(\frac{1}{3}\right) + \frac{2125}{12096}\left(\frac{2}{3}\right) + \frac{250}{567}(1) - \frac{3875}{12096}\left(\frac{4}{3}\right) + \frac{235}{1512}\left(\frac{5}{3}\right) - \frac{275}{36288}(2)\right] \\
\sum_{j=0}^{\infty} \frac{(2)^j}{j!} - y_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left[\frac{18}{35}\left(\frac{1}{3}\right) + \frac{9}{140}\left(\frac{2}{3}\right) + \frac{68}{105}(1) - \frac{9}{140}\left(\frac{4}{3}\right) + \frac{18}{35}\left(\frac{5}{3}\right) - \frac{41}{420}(2)\right]
\end{array}\right]$$

Therefore, according to [13], the new method is of uniform order seven as well as error constant is given by

$$C_7 = \begin{pmatrix} 1.5460 \times 10^{-06} \\ 1.1321 \times 10^{-06} \\ 1.3677 \times 10^{-06} \\ 1.1218 \times 10^{-06} \\ 1.5926 \times 10^{-06} \\ 1.2661 \times 10^{-06} \end{pmatrix}$$

3.2 Consistent

Traditionally, the new method is consistent because the order of the method is order greater than or equal to one [3].

3.3 Zero Stable

By definition, the new method is said to be zero stable as $h \to 0$ if the roots of the polynomial $\pi(r) = 0$ satisfy $\left|\left[\sum A^0 R^{k-1}\right]\right| \leq 1$, and those roots with R = 1 must be simple. Hence it's found as

$$\pi(r) = \left| r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} r & 0 & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & 0 & -1 \\ 0 & 0 & r & 0 & 0 & -1 \\ 0 & 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & 0 & r-1 \end{bmatrix} \right| = r^6 (r-1)$$

Then, solving for r in $r^6(r-1)$,

gives $r = 0, 0, 0, 0, 0, 1$. Therefore, the method is zero stable [16].

Dahlquist's theorem states that, the new method is convergent and consistency and zero-stability are analyzed and fulfilled.

3.4 Convergence

Theorem 3.1

Consistency and zero-stability are both required and sufficient conditions for a linear multistep method to be convergent. Therefore, the new method is convergent since it is consistent and zero-stable [15].

3.5 Region of Absolute Stability

The boundary locus method is used to generate the new method's stability polynomial [15]. The polynomial is defined as

$$\left.\begin{array}{l}\bar{h}(w)=\left(-\frac{1}{5103}w^5+\frac{1}{5103}w^6\right)h^6+\left(-\frac{7}{2430}w^5-\frac{7}{2430}w^6\right)h^5+\left(-\frac{29}{1215}w^5+\frac{29}{1215}w^6\right)h^4\\ +\left(-\frac{7}{54}w^5-\frac{7}{54}w^6\right)h^3+\left(-\frac{25}{54}w^5+\frac{25}{54}w^6\right)h^2+\left(w^5+w^6\right)h+w^5+w^6\end{array}\right\} \quad (3.2)$$

The polynomial is used to plot the region as

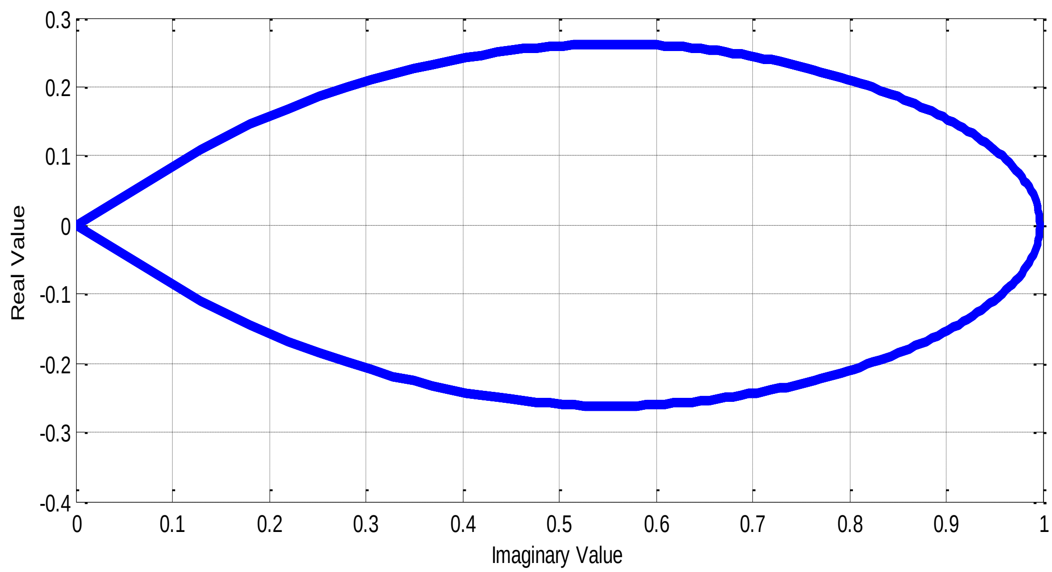

Fig. 3.1: Showing the region of stability of new method.

4 Results and Discussions

This section presents and discusses the results derived from various numerical examples. Additionally, the effectiveness of the proposed method is evaluated using four real-world problems, including the Malthusian Growth Model, the Prothero-Robinson equation, and other highly stiff oscillatory differential equations. For each case, the approximate solutions are compared to numerical benchmarks, and the absolute errors from the new method are contrasted with those from existing approaches to assess its accuracy and performance.

4.2 Numerical Examples

To evaluate the effectiveness of the developed methods, we present several numerical examples, including the following five cases **Example 4.1: Malthus Growth Model**

The Malthusian growth model, introduced by British economist Thomas Robert Malthus, is a population growth theory that highlights the potential for exponential population increase and its impact on resource availability. Malthus first articulated this theory in his 1798 work, *An Essay on the Principle of Population.* The model suggests that populations grow exponentially when resources are plentiful, leading to a scenario where the population size can double at a consistent rate over time, assuming no limiting factors. In contrast, Malthus argued that food supply increases arithmetically, by a fixed amount each period, resulting in a mismatch between population growth and resource availability. According to Malthus, this imbalance ultimately leads to natural checks such as famine, war, or disease, which reduce the population. (Oluwaseun and Zurni (2022).

The Malthusian growth model can be described using a simple differential equation:

$$\frac{dy}{dx} = kp,\ x \in [0,1], \tag{4.1}$$

with the exact solution given by

$$y(x) = 100\exp(0.250679566129x) \tag{4.2}$$

Initial condition $y(0) = 100$ with $k = 0.250679566129$ and $h = 0.1$

Source: [7, 8].

Example 4.2: **(Prothero-Robinson Equation)**

Take into account the Prothero-Robinson oscillatory differential equation, which has been addressed by [16, 17], formulated as follows:

$$y' = \Phi(y - \sin x) - y,\ \Phi = -1,\ y(0) = 0 \tag{4.3}$$

Which has the exact solution as

$$y(x) = \sin x \tag{4.4}$$

Example 4.3: Consider the differential equation

$$\frac{du}{dv} = -\sin(v) - 200(u - \cos(v)),\ h = 0.01,\ u(0) = 0 \tag{4.5}$$

with the exact solution

$$u(v) = \cos(v) - e^{-200v} \tag{4.6}$$

Source: [17, 18]

Example 4.4: Consider the oscillatory differential equation

$$\frac{du}{dv} = -10(u-1)^2,\ h = 0.01,\ u(0) = 2 \tag{4.7}$$

with the exact solution

$$u(v) = 1 + \frac{1}{1+10v} \tag{4.8}$$

Source: [18, 19]

Example 4.5: Consider the Highly stiff oscillatory differential equation

$$\frac{du}{dv} = -\psi u,\ h = 0.1,\ u(0) = \psi = 1 \tag{4.9}$$

with the exact solution

$$u(v) = \exp(-v) \tag{4.10}$$

Source: [20, 21)]

Table 4.1: The results of example 4.1 with [7, 8]

x	Exact Solution	Computed Solution	Absolute Errors	Error in [7]	Error in [8]
0.100	102.53847998347329794000	102.53847998347329790000	4.0000(-17)	1.6677(-08)	0.0000(00)
0.200	105.14139877321154182000	105.14139877321154182000	0.0000(00)	4.4003(-10)	0.0000(00)
0.300	107.81039213541335645000	107.81039213541335642000	3.0000(-17)	1.7117(-08)	0.0000(00)
0.400	110.54713735987489512000	110.54713735987489512000	0.0000(00)	8.8005(-10)	0.0000(00)
0.500	113.35335431405805132000	113.35335431405805129000	3.0000(-17)	1.7557(-08)	1.4211(-14)
0.600	116.23080652391598100000	116.23080652391598099000	1.0000(-17)	1.3201(-09)	1.4211(-14)
0.700	119.18130228215516429000	119.18130228215516425000	4.0000(-17)	1.7997(-08)	1.4211(-14)
0.800	122.20669578463047796000	122.20669578463047795000	1.0000(-17)	1.7601(-09)	0.0000(00)
0.900	125.30888829558742918000	125.30888829558742917000	1.0000(-17)	1.8437(-08)	1.4211(-14)
1.000	128.48982934248383035000	128.48982934248383034000	1.0000(-17)	2.2001(-09)	0.0000(00)

Table 4.2: The results of application problem 4.2 with [16, 17]

x	Exact Solution	Computed Solution	Absolute Errors	Error in [16]	Error in [17]
0.100	0.09983341664682815231	0.09983341664682691151	1.2408(-15)	1.4530(-11)	1.3422(-11)
0.200	0.19866933079506121546	0.19866933079506177666	5.6120(-16)	1.6211(-11)	2.1464(-11)
0.300	0.29552020666133957511	0.29552020666133611456	3.4606(-15)	2.1310(-11)	3.2359(-11)
0.400	0.38941834230865049167	0.38941834230865202176	1.5301(-15)	1.3799(-11)	4.1877(-11)
0.500	0.47942553860420300027	0.47942553860419784705	5.1532(-15)	2.7441(-11)	4.6377(-11)
0.600	0.56464247339503535720	0.56464247339503814732	2.7901(-15)	1.1114(-11)	5.3368(-11)
0.700	0.64421768723769105367	0.64421768723768473182	6.3219(-15)	2.8657(-11)	5.8936(-11)
0.800	0.71735609089952276163	0.71735609089952698884	4.2272(-15)	1.9218(-10)	6.0221(-11)
0.900	0.78332690962748338846	0.78332690962747641090	6.9776(-15)	1.2392(-10)	6.3342(-11)
1.000	0.84147098480789650665	0.84147098480790223848	5.7318(-15)	1.4711(-10)	6.5059(-11)

Table 4.3: The results of application problem 4.3 with [17, 18]

x	Exact Solution	Computed Solution	Absolute Errors	Error in [17]	Error in [18]
0.001	0.18126874692477177712	0.18126874692205980800	2.7120(-12)	6.5812(-06)	3.7249(-10)
0.002	0.32967795396412439246	0.32967795396502736584	9.0297(-13)	2.9379(-06)	5.2169(-10)
0.003	0.45118386391042716158	0.45118386390934856636	1.0786(-12)	9.3961(-05)	6.7870(-10)
0.004	0.55066303589223450724	0.55066303589344506955	1.2106(-12)	1.1305(-05)	7.6010(-10)
0.005	0.63210805885482676508	0.63210805885459932337	2.2744(-12)	7.9107(-06)	7.4126(-10)
0.006	0.69878778814058064233	0.69878778814179783856	1.2172(-12)	1.0313(-05)	7.4495(-10)
0.007	0.75337853615825529977	0.75337853615843502633	1.7973(-13)	1.0426(-05)	7.2211(-10)
0.008	0.79807148217492301264	0.79807148217601089409	1.0879(-12)	7.7981(-05)	6.5649(-10)
0.009	0.83466061205144457875	0.83466061205178772359	3.4315(-13)	8.4900(-05)	6.1326(-10)
0.01	0.86461471717914105002	0.86461471718005258589	9.1150(-13)	8.0388(-05)	5.6367(-10)

Table 4.4: The results of application problem 4.4 with [18, 19]

x	Exact Solution	Computed Solution	Absolute Errors	Error in [18]	Error in [19]
0.001	1.90909090884750640830	1.90909090889090909090	4.3403(-11)	2.4025(-08)	1.0700(-05)
0.002	1.83333333337241953740	1.83333333333333333330	3.9086(-11)	3.1560(-08)	2.3800(-05)
0.003	1.76923076920944483900	1.76923076923076923080	2.1324(-11)	3.2631(-08)	4.5100(-05)
0.004	1.71428571432193859870	1.71428571428571428570	3.6224(-11)	3.1192(-08)	6.2000(-04)
0.005	1.66666666668304290430	1.66666666666666666670	1.6376(-11)	2.8877(-08)	8.8400(-04)
0.006	1.62500000002955801560	1.62500000000000000000	2.9558(-11)	2.6370(-08)	1.0300(-03)
0.007	1.58823529413888054590	1.58823529411764705880	2.1234(-11)	2.3953(-08)	1.2700(-03)
0.008	1.55555555557943834040	1.55555555555555555560	2.3883(-11)	2.1734(-08)	1.5300(-03)
0.009	1.52631578949329163390	1.52631578947368421050	1.9607(-11)	1.9740(-08)	1.7500(-03)
0.010	1.50000000001952055900	1.50000000000000000000	1.9521(-11)	1.7969(-08)	1.8100(-03)

Table 4.5: The results of application problem 4.5 with [20, 21]

x	Exact Solution	Computed Solution	Absolute Errors	Error in [20]	Error in [21]
0.100	0.90483741803595957316	0.90483741803596084590	1.2727(-15)	9.0730(-12)	5.0000(-10)
0.200	0.81873075307798185867	0.81873075307798400161	2.1429(-15)	1.1768(-11)	5.0000(-10)
0.300	0.74081822068171786607	0.74081822068170938478	8.4813(-15)	2.3144(-11)	8.0000(-10)
0.400	0.67032004603563930074	0.67032004603564280973	3.5090(-15)	2.8440(-11)	7.0000(-10)
0.500	0.60653065971263342360	0.60653065971262806724	5.3564(-15)	3.1815(-11)	1.1000(-09)
0.600	0.54881163609402643263	0.54881163609403074200	4.3094(-15)	3.4927(-11)	1.1000(-09)
0.700	0.49658530379140951470	0.49658530379140642905	3.0857(-15)	3.6582e-11	1.1000(-09)
0.800	0.44932896411722159143	0.44932896411722629572	4.7043(-15)	3.8127(-11)	1.0000e-09
0.900	0.40656965974059911188	0.40656965974059764972	1.4622(-15)	3.8576(-11)	1.0000e-09
1.000	0.36787944117144232160	0.36787944117144713602	4.8144(-15)	3.9020(-11)	1.0000(-09)

4.3 Discussion and conclusion

This study introduces the use power series polynomial to derive the new method for solving various real-life problems in form of first-order stiff initial value problems. The new method were focused on their basic properties such as order, error constant, consistency, zero-stability and stability regions. The methods were applied to real-life problems, and results from tables 4.1 to 4.5. In this study, we have applied the new method on five numerical examples. Example 4.1 is the Malthus growth model and the results are presented in Table 4.1. These results are compared with those of [7, 8]. It is evident that the new method perform better than the methods proposed by [7, 8]. Example 4.2 involves the Prothero differential equation, which was analyzed using the new method. The comparisons of the results are shown in Table 4.2, alongside the solutions provided by [16, 17]. According to Table 4.2, the new method exhibit better convergence than the methods of [16, 17]. When solving the oscillatory differential equation in Example 4.3, the new method demonstrates faster convergence compared to the existing methods of [17, 18] for similar examples. For the oscillatory differential equation in Example 4.4, the new method outperform the methods of [18, 19] when solving similar examples, as shown in Table 4.4. Finally, Example 4.5 deals with another oscillatory differential equation. The results of solving Example 4.5 using the methods of [20, 21] are displayed in Table 4.5.

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Appendix

$$\psi_{01},\psi_{02},\psi_{03},\psi_{04},\psi_{05},\psi_{06},\psi_{07}=-\frac{19087h}{181440},-\frac{2713h}{7560},\frac{15487h}{60580},-\frac{586h}{2835},\frac{6737h}{60480},-\frac{263h}{7560},\frac{863h}{181440}$$

$$\psi_{11},\psi_{12},\psi_{13},\psi_{14},\psi_{15},\psi_{16},\psi_{17}=-\frac{863h}{181440},-\frac{349h}{2520},\frac{5221h}{20160},-\frac{254}{2835},\frac{811h}{20160},-\frac{29h}{2520},\frac{271h}{181440}$$

$$\psi_{21},\psi_{22},\psi_{23},\psi_{24},\psi_{25},\psi_{26},\psi_{27}=-\frac{37h}{11340},\frac{233h}{1890},\frac{1621h}{3780},\frac{322h}{2835},\frac{11h}{3780},-\frac{h}{378},\frac{h}{2268}$$

$$\psi_{31},\psi_{32},\psi_{33},\psi_{34},\psi_{35},\psi_{36},\psi_{37}=-\frac{29h}{6720},\frac{37h}{280},\frac{877h}{2240},\frac{34h}{105},\frac{387h}{2240},-\frac{h}{56},\frac{13h}{6720}$$

$$\psi_{41},\psi_{42},\psi_{43},\psi_{44},\psi_{45},\psi_{46},\psi_{47}=-\frac{8h}{2835},\frac{38h}{315},\frac{136h}{315},\frac{664h}{2835},\frac{136h}{315},-\frac{38h}{315},-\frac{8h}{2835}$$

$$\psi_{51},\psi_{52},\psi_{53},\psi_{54},\psi_{55},\psi_{56},\psi_{57}=-\frac{275h}{36288},\frac{235h}{1512},\frac{3875h}{12096},\frac{250h}{567},\frac{2125h}{12096},-\frac{725h}{1512},-\frac{3715h}{36288}$$

$$\Omega_{11},\Omega_{12},\Omega_{13},\Omega_{14},\Omega_{15},\Omega_{16},\Omega_{17}=\frac{19087h}{181440},\frac{2713h}{7560},-\frac{15487h}{60480},\frac{586h}{2835},-\frac{6737h}{60480},\frac{263h}{7560},-\frac{863h}{2835}$$

$$\Omega_{21},\Omega_{22},\Omega_{23},\Omega_{24},\Omega_{25},\Omega_{26},\Omega_{27}=\frac{1139h}{11340},\frac{94h}{189},\frac{11h}{3780},\frac{332h}{2835},-\frac{269h}{3780},\frac{22h}{945},-\frac{37h}{11340}$$

$$\Omega_{31},\Omega_{32},\Omega_{33},\Omega_{34},\Omega_{35},\Omega_{36},\Omega_{37}=\frac{137h}{1344},\frac{27h}{56},\frac{387h}{2240},\frac{34h}{105},-\frac{243h}{2240},\frac{9h}{280},-\frac{29h}{6720}$$

$$\Omega_{41},\Omega_{42},\Omega_{43},\Omega_{44},\Omega_{45},\Omega_{46},\Omega_{47}=\frac{286h}{2835},\frac{464h}{945},\frac{128h}{945},\frac{1504h}{2835},\frac{58h}{945},\frac{16h}{945},-\frac{8h}{2835}$$

$$\Omega_{51},\Omega_{52},\Omega_{53},\Omega_{54},\Omega_{55},\Omega_{56},\Omega_{57}=\frac{3715h}{36288},\frac{725h}{1512},\frac{2125h}{12096},\frac{250h}{567},\frac{3875h}{12096},\frac{235h}{1512},-\frac{275h}{3628}$$

$$\Omega_{61},\Omega_{62},\Omega_{63},\Omega_{64},\Omega_{65},\Omega_{66},\Omega_{67}=\frac{41h}{420},\frac{18h}{35},\frac{9h}{140},\frac{68h}{105},\frac{9h}{140},\frac{18h}{35},-\frac{41h}{420}$$